Bivariate Analysis of Incomplete Degradation Observations Based on Inverse Gaussian Processes and Copulas

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***Abstract*—Modern engineering systems are generally composed of multicomponents and are characterized as multifunctional. Condition monitoring and health management of these systems often confronts the difficulty of degradation analysis with multiple performance characteristics. Degradation observations generally exhibit an s-dependent nature and sometimes experience incomplete measurements. These issues necessitate investigating multiple s-dependent degradations analysis with incomplete observations. In this paper, a new type of bivariate degradation model based on inverse Gaussian processes and copulas is proposed. A two-stage Bayesian method is introduced to implement parameter estimation for the bivariate degradation model by treating the degradation processes and copula function separately. Degradation inferences for missing observation points, and for future observation points are investigated. A simulation study is presented to study the effectiveness of the dependence modeling and degradation inference of the proposed method. For demonstration, a bivariate degradation analysis of positioning accuracy and output power of heavy machine tools subject to incomplete measurements is provided.**

***Index Terms*—Bayesian reliability, bivariate degradation process, copula function, degradation analysis, inverse Gaussian (IG) process.**

Abbreviation and Acronyms

PM	Preventive maintenance.
CM	Condition monitoring.
SHM	System health management.
IG	Inverse Gaussian.
PDF	Probability density function.
CDF	Cumulative distribution function.
RUL	Remaining useful life.
MCMC	Markov chain Monte Carlo method.

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Notation

$Y(t)$	Degradation process.
$\Lambda(t)$	Mean function.
λ	Scale parameter.
$\Delta Y(t)$	Degradation increment.
$\mathrm{IG}(a,b)$	Inverse Gaussian distribution.
$f(y\|a,b)$	PDF of an inverse Gaussian distribution.
$F(y\|a,b)$	CDF of an inverse Gaussian distribution
T	Lifetime.
$\Pr\{A\}$	Probability of an event A.
$F(t)$	CDF of lifetime T.
$C(u,v)$	Bivariate copula function.
$F(y_1,y_2)$	Bivariate distribution function.
n	Number of observed samples.
i	Index of observed samples with $i=1,\ldots,n$.
m_i	Number of observation points of the ith sample.
j	Index of observation point of the ith sample with $j=1,\ldots,m_i$.
k	Index of bivariate degradation process with $k=1,2$.
$Y_k(t)$	kth degradation process.
$y_k(t_{i,j})$	Degradation observation of the kth degradation process for the ith sample at the jth observation time point.
Δy_{kij}	Degradation increment of the kth degradation process for the ith sample between the jth and the $(j-1)$th observation time points.
$F_k(\Delta y_{kij})$	CDF of degradation increment Δy_{kij}.
$F(\Delta y_{1ij},\Delta y_{2ij})$	Joint CDF of Δy_{1ij} and Δy_{2ij}.
D_k	Degradation threshold of the kth degradation process.
$R(t)$	Reliability function.

T^{RUL}	Remaining useful life.
$(y_1(t_{i,j}), y_2(t_{i,j}))$	Complete bivariate degradation observation.
$(\cdot, y_2(t_{i,h_i+1}))$	Incomplete bivariate degradation observation.
$\mathbf{Y}_i^{1:m_i}$	Set of complete bivariate degradation observations of the ith sample.
$\mathbf{Y}_i^{1:h_i}$	Set of incomplete bivariate degradation observations of the ith sample.
$\Delta\mathbf{y}_k$	Set of degradation increments of the kth degradation process.
$\boldsymbol{\theta}$	Parameters of the bivariate degradation model.
$\boldsymbol{\theta}_{\mathrm{IG}k}$	Parameters of the kth degradation process.
$\boldsymbol{\theta}_{\mathrm{C}}$	Parameters of copula function.
$\hat{\boldsymbol{\theta}}$	Point estimation of $\boldsymbol{\theta}$.
$p(\boldsymbol{\theta}\|\mathbf{Y})$	Posterior distribution of $\boldsymbol{\theta}$.
$\pi(\boldsymbol{\theta})$	Prior distribution of $\boldsymbol{\theta}$.
$F(u\|v)$	Conditional distribution of u given v.
$c(u, v)$	Density of a bivariate copula function.
$C(u\|v, \boldsymbol{\theta}_{\mathrm{C}})$	Conditional copula function.

I. Introduction

Complex systems are indispensable factors in modern society, including manufacturing systems, commercial airplanes, and high-speed trains. The reliability of these systems has become a critical issue both for the desire of high availability and for the pursuit of high safety [1], [2]. Government and industry are relying more and more heavily on advanced methods to determine reliability of complex systems [3], [4]. Methods such as condition monitoring (CM) [5] and degradation analysis [6] have been developed to facilitate the reliability assessment [7], preventive maintenance (PM) [8], and system health management (SHM) [9] of these systems.

Traditional methods for CM and degradation analysis of complex systems assume that only one performance indicator is monitored, and the system experiences a failure when this performance indicator reaches a predefined threshold. However, a complex system may possess multiple functions and may have multiple performance indicators [10]. The methods based on one performance indicator cannot be applied to situations with multiple performance indicators. A classic example for this point, which motivates the research presented in this paper, is the degradation analysis of heavy machine tools. The positioning accuracy and output power are two indispensable performance indicators of heavy machine tools. The differences of these indicators and their measurement techniques lead to different types of degradation observations, which are characterized as bivariate s-dependent degradation processes with incomplete degradation observations. A suitable degradation modeling and a precise degradation analysis of these two degradation processes are critical for the operation and management of heavy machine tools.

In the past decades, degradation modeling has been studied a great deal. There are generally four types of degradation models: degradation path models [11], regression-based models [12], Markov chain-based models [13], and stochastic process-based models [14]. Most of these works use a single degradation process model. To facilitate the degradation analysis of complex systems with multiple performance indicators, Wang and Coit [15] investigated the degradation analysis of a system with multiple degradation measurements, where a multivariate s-normal distribution based model was introduced. Sari *et al.* [16] studied the reliability assessment of light-emitting diodes considering the dependence between two performance indicators. A generalized linear model and a copula function were used to construct a bivariate degradation process model in their paper. Recently, bivariate degradation process models based on gamma processes, Wiener processes and copula functions were introduced by Pan and Balakrishnan [17], Pan *et al.* [18], Wang *et al.* [19], and Wang *et al.* [20].

Among these literatures, the fatigue crack data provided by Meeker and Escobar [11] were used as numerical examples for most of the bivariate degradation process models, although these data were originally introduced for one-dimensional (1-D) degradation modeling. This dataset cannot be used to represent the problem of degradation analysis with different performance indicators subject to incomplete degradation observations. However, the problem of reliability analysis with incomplete observations is common and critical for complex systems, which has been highlighted and investigated by Zhang *et al.* [21], Ye *et al.* [22], Ye *et al.* [23], and Si *et al.* [24]. Unfortunately, these studies mainly focused on the degradation analysis with only 1-D performance indicator. Bivariate degradation analysis with incomplete observations has not been studied thoroughly, especially for the situation where some degradation observations are missing. Accordingly, a flexible method for parameter estimation with incomplete observations, and a feasible method for degradation inferences of missing observations are needed. Both methods are crucial for bivariate degradation analysis and for the follow-up PM and SHM of complex systems.

In addition, most of the proposed bivariate degradation process models are based on classic stochastic processes, including the Wiener process and the gamma process. Recently, the inverse Gaussian (IG) process was introduced as a flexible degradation process for degradation modeling by Wang and Xu [25], Zhang *et al.* [21], Ye and Chen [26], Peng [27], and Peng *et al.* [28]. It has been demonstrated that the IG process is more suitable than the Wiener process and the gamma process for degradation modeling in some applications. It is therefore of interest to further study bivariate degradation analysis based on the IG process models.

Based on the motivation and literature review presented above, this paper is aimed to deliver three contributions.

- We introduce a new type of bivariate degradation process model based on the IG processes and copula functions.
- We present a two-stage Bayesian parameter estimation method to cope with the parameter estimations for complete and incomplete degradation observations.

- We propose two degradation inference strategies separately for the degradation inferences of missing degradation observations, and for the degradation inferences of future degradation observations. A method for remaining useful life (RUL) prediction based on the missing degradation observations is also provided.

In addition, a simulation study is presented to verify the capability of the proposed method. An illustrative example originating from a practical engineering project dealing with degradation analysis of heavy machine tools is presented to illustrate the proposed method. This illustrative example is characterized as two different performance indicators with incomplete degradation observations.

The remainder of this paper is organized as follows. Section II introduces the bivariate degradation process model with brief descriptions of the IG degradation process and the copula functions. Section III presents the bivariate degradation analysis with incomplete degradation observations. The two-stage parameter estimation method and the two degradation inference strategies are presented. Section IV describes the simulation study of the proposed method, where the capability of degradation inference is verified. Section V presents the degradation analysis of heavy machine tools to illustrate the proposed method. Section VI concludes this paper with several points for future research.

II. Bivariate Degradation Model Based on IG Processes and Copulas

A. IG Process Model

A simple IG process $\{Y(t), t \geq 0\}$ with mean function $\Lambda(t)$ and scale parameter λ is denoted as $Y(t) \sim \mathrm{IG}(\Lambda(t), \lambda\Lambda(t)^2)$ [25]. The IG process has the following properties: $Y(t) \in [0, +\infty)$ with $Y(0) = 0$, and the degradation increments $\Delta Y(t) = Y(t+\Delta t) - Y(t)$ on disjoint intervals are s-independent, and follow IG distributions as $\Delta Y(t) \sim \mathrm{IG}(\Delta\Lambda(t), \lambda\Delta\Lambda(t)^2)$ with $\Delta\Lambda(t) = \Lambda(t+\Delta t) - \Lambda(t)$.

The mean and variance of $Y(t)$ are $\Lambda(t)$ and $\Lambda(t)/\lambda$, where $\Lambda(t)$ is a monotone increasing function with $\Lambda(0) = 0$, and also an approximate description of $Y(t)$. Classical forms of $\Lambda(t)$ include a power-law function, an exponential function, and a physical-model based function [28].

The probability density function (PDF) and the cumulative distribution function (CDF) of an IG distribution for $y \sim \mathrm{IG}(a, b), a, b > 0$ with mean a and variance a^3/b are

$$f(y|a,b) = \sqrt{\frac{b}{2\pi y^3}} \exp\left(-\frac{b\,(y-a)^2}{2a^2 y}\right), \quad y > 0 \quad (1)$$

$$F(y|a,b) = \Phi\left(\sqrt{\frac{b}{y}}\left(\frac{y}{a} - 1\right)\right) + \exp\left(\frac{2b}{a}\right)\Phi\left(-\sqrt{\frac{b}{y}}\left(\frac{y}{a} + 1\right)\right) \quad (2)$$

where $\Phi(\cdot)$ is the standard s-normal CDF.

The PDFs and CDFs of $Y(t)$ and $\Delta Y(t)$ can be obtained based on (1) and (2).

Suppose a product is observed with a degradation process characterized as $\{Y(t), t \geq 0\}$. The product fails when the degradation process first reaches a predefined threshold D. Accordingly, the lifetime T of the product is defined as $T = \inf\{t|Y(t) \geq D\}$. The CDF of the lifetime T is then obtained as

$$\begin{aligned} F(t) &= \Pr\{T \leq t\} = \Pr\left\{Y(t) \geq D|\Lambda(t), \lambda\Lambda(t)^2\right\} \\ &= \Phi\left(\sqrt{\frac{\lambda}{D}}\,(\Lambda(t) - D)\right) \\ &\quad - \exp(2\lambda\Lambda(t))\,\Phi\left(-\sqrt{\frac{\lambda}{D}}\,(\Lambda(t) + D)\right). \end{aligned} \quad (3)$$

B. Copula Function

The copula function was introduced by Sklar [29], [30] to model the dependence of a group of random variables. By adopting a copula function, the dependence structure of the random variables can be characterized separately from their marginal distribution functions. In this study, we focus on bivariate copula functions. A bivariate copula function is a joint CDF of two uniformly distributed random variables on the interval $[0, 1]$ as

$$C(u,v) = \Pr\{U \leq u, V \leq v\} = F_{UV}(u,v) \quad (4)$$

where $F_{UV}(u,v)$ is the joint CDF of uniformly distributed random variables U and V.

Let Y_1 and Y_2 denote two random variables with marginal CDFs as $F_1(y_1)$ and $F_2(y_2)$. According to Sklar's theory [30], a bivariate distribution function $F(y_1, y_2)$ for Y_1 and Y_2 is constructed through a bivariate copula function as

$$\begin{aligned} F(y_1, y_2) &= \Pr\{Y_1 \leq y_1, Y_2 \leq y_2\} \\ &= \Pr\{U \leq F_1(y_1), V \leq F_2(y_2)\} \\ &= C(F_1(y_1), F_2(y_2)). \end{aligned} \quad (5)$$

The dependence of these two random variables is characterized by $C(u,v)$. By using different copula functions, different types of bivariate distribution functions can be constructed. Popular bivariate copulas include the following: the Gaussian copula, the Frank copula, the Gumbel copula, and the Clayton copula [16], [31]. For more information about copula functions, please refer to the work by Nelsen [30] and the works presented in [16] and [31]–[34].

C. Bivariate Degradation Model

Assume that a product has been observed with two s-dependent degradation processes. These degradation processes are characterized as the IG processes: $\{Y_1(t), t \geq 0\}$, and $\{Y_2(t), t \geq 0\}$. Suppose n samples of the product are observed, and each sample is observed at m_i different observation time points with $i = 1, \ldots, n$. Let $y_k(t_{i,j})$ denote the jth observation of the degradation process k for sample i at time point $t_{i,j}$, where $j = 1, \ldots, m_i$ and $k = 1, 2$. Let $\Delta y_{kij} = y_k(t_{i,j}) - y_k(t_{i,j-1})$ denote the degradation increment with $y_k(t_{i,0}) = 0$ and $t_{i,0} = 0$.

The dependence between $Y_1(t)$ and $Y_2(t)$ is constructed as follows. We assume that, for the ith sample, the $y_1(t_{i,j}) - y_1(t_{i,j-1})$

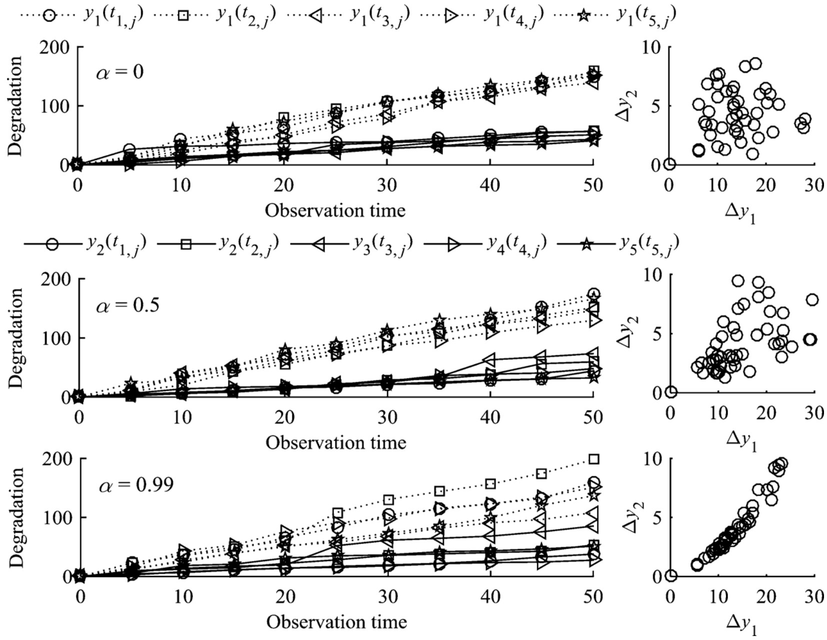

Fig. 1. Plots of a bivariate degradation process based on the IG processes and the Gaussian copula with different linear coefficients.

and $y_2(t_{i,j}) - y_2(t_{i,j-1})$ in the same time interval $[t_{i,j-1}, t_{i,j}]$ are s-dependent, whereas the degradation increments in disjoint time intervals are s-independent, e.g., $y_1(t_{i,j-1}) - y_1(t_{i,j-2})$ and $y_2(t_{i,j}) - y_2(t_{i,j-1})$ in $[t_{i,j-2}, t_{i,j-1}]$ and $[t_{i,j-1}, t_{i,j}]$ are s-independent [18], [19]. In addition, we further assume that, for different samples, the degradation observations are s-independent, e.g., $y_1(t_{i,j}) - y_1(t_{i,j-1})$ and $y_2(t_{i+1,j}) - y_2(t_{i+1,j-1})$ from the ith sample and the $(i+1)$th sample are s-independent, regardless of whether $[t_{i,j-1}, t_{i,j}]$ and $[t_{i+1,j-1}, t_{i+1,j}]$ are the same or disjoint. We have two marginal CDFs, $F_1(\Delta y_{1ij})$ and $F_2(\Delta y_{2ij})$, respectively for Δy_{1ij} and Δy_{2ij} with $\Delta y_{1ij} \sim \mathrm{IG}(\Delta\Lambda_1(t_{ij}), \lambda_1 \Delta\Lambda_1(t_{ij})^2)$ and $\Delta y_{2ij} \sim \mathrm{IG}(\Delta\Lambda_2(t_{ij}), \lambda_2 \Delta\Lambda_2(t_{ij})^2)$. To model the dependence of Δy_{1ij} and Δy_{2ij}, a copula function is used to construct their joint CDF as

$$F\left(\Delta y_{1ij}, \Delta y_{2ij}\right) = C\left(F_1\left(\Delta y_{1ij}\right), F_2\left(\Delta y_{2ij}\right)\right). \quad (6)$$

Given the marginal CDFs and joint CDF of the degradation increments for $Y_1(t)$ and $Y_2(t)$, the bivariate degradation model is then constructed as

$$\begin{aligned} y_1(t_{i,j}) &= \sum\nolimits_{l=1}^{j} \Delta y_{1il}, y_2(t_{i,j}) \\ &= \sum\nolimits_{l=1}^{j} \Delta y_{2il}, l = 1, \ldots, j \\ \Delta y_{1il} &\sim \mathrm{IG}\left(\Delta\Lambda_1\left(t_{i,l}\right), \lambda_1 \Delta\Lambda_1\left(t_{i,l}\right)^2\right), \ldots, \\ \Delta y_{2il} &\sim \mathrm{IG}\left(\Delta\Lambda_2\left(t_{i,l}\right), \lambda_2 \Delta\Lambda_2\left(t_{i,l}\right)^2\right) \\ F\left(\Delta y_{1il}, \Delta y_{2il}\right) &= C\left(F_1\left(\Delta y_{1il}\right), F_2\left(\Delta y_{2il}\right)\right). \end{aligned} \quad (7)$$

Without loss of generality, a pictorial description of a bivariate degradation process is shown in Fig. 1 to demonstrate the model proposed in (7). The IG processes $Y_1(t) \sim \mathrm{IG}(\mu_1 t, \lambda_1 (\mu_1 t)^2)$, and $Y_2(t) \sim \mathrm{IG}(\mu_2 t, \lambda_2 (\mu_2 t)^2)$ with $\mu_1 = 3, \mu_2 = 1, \lambda_1 = \lambda_2 = 0.35$, and the Gaussian copula with linear coefficients $\alpha = 0, 0.5, 0.99$ are used. The specific model is given as

$$\begin{aligned} \Delta y_{1ij} &\sim \mathrm{IG}\left(\mu_1 t_{i,j}, \lambda_1 \left(\mu_1 t_{i,j}\right)^2\right) \\ \Delta y_{2ij} &\sim \mathrm{IG}\left(\mu_2 t_{i,j}, \lambda_2 \left(\mu_2 t_{i,j}\right)^2\right) \\ F\left(\Delta y_{1ij}, \Delta y_{2ij}\right) &= \int_{-\infty}^{\Phi^{-1}(F_1(\Delta y_{1ij}))} \int_{-\infty}^{\Phi^{-1}(F_2(\Delta y_{2ij}))} \frac{1}{2\pi\sqrt{1-\alpha^2}} \\ &\quad \times \exp\left(-\frac{x^2 - 2\alpha xy + y^2}{2\left(1-\alpha^2\right)}\right) \mathrm{d}x \mathrm{d}y \end{aligned} \quad (8)$$

where $\Phi^{-1}(\cdot)$ is the inverse CDF of a standard s-normal distribution.

Further assume that the product fails when either of $Y_1(t)$ and $Y_2(t)$ crosses their respective degradation thresholds D_1 and D_2 [20]. The reliability function of the product is then given as

$$R(t) = \Pr\left\{Y_1(t) < D_1, Y_2(t) < D_2\right\}. \quad (9)$$

If the degradation processes are observed up to time point t_{i,m_i} for the ith sample, the RUL of this sample is expressed as (10), shown at the bottom of the page, where $\mathbf{y}_{1,i,1:m_i}$ and $\mathbf{y}_{2,i,1:m_i}$ separately denote the observations of $Y_1(t)$ and $Y_2(t)$ up to the time point t_{i,m_i}.

$$T^{\mathrm{RUL}} = \inf\left\{ r : \begin{array}{c} Y_1\left(t_{i,m_i} + r\right) \geq D_1 \\ \text{or} \\ Y_2\left(t_{i,m_i} + r\right) \geq D_2 \end{array} \middle| \begin{array}{c} \mathbf{y}_{1,i,1:m_i}, Y_1\left(t_{i,m_i}\right) < D_1 \\ \text{and} \\ \mathbf{y}_{2,i,1:m_i}, Y_2\left(t_{i,m_i}\right) < D_2 \end{array} \right\} \quad (10)$$

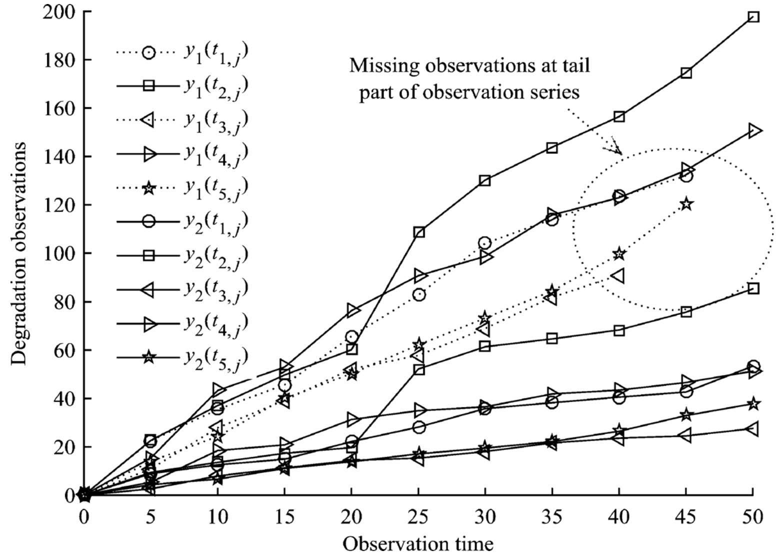

Fig. 2. Example of incomplete bivariate degradation observations.

III. Bivariate Degradation Analysis With Incomplete Observations

There are three indispensable aspects for a coherent degradation analysis: parameter estimation, degradation inference, and RUL prediction. In this section, a two-stage Bayesian method is introduced to facilitate the parameter estimation. Two degradation inference strategies are introduced for the degradation inferences of missing degradation observations and future degradation observations. A method for the RUL prediction based on parameter estimation and degradation inferences is also provided.

A. Bivariate Incomplete Degradation Observations

In this paper, we mainly focus on the bivariate degradation process with missing observations at the tail part of the observation series. This kind of incomplete degradation observations is presented as follows: one degradation process is observed with complete observations, and the other is observed with incomplete observations at the tail part of the observation series. A pictorial description of the incomplete degradation observations is presented in Fig. 2, where $Y_2(t)$ is observed completely, whereas the observations of $Y_1(t)$ are missing at the tail part of the observation series for some samples. Degradation observations with other types of missing observations are not discussed in this paper, because other types of missing observations, such as missing observations in the intermediate part of the observation series, do not directly affect the degradation inference and RUL prediction.

Let $\mathbf{Y}_i^{1:m_i} = \{(y_1(t_{i,1}), y_2(t_{i,1})), \ldots, (y_1(t_{i,m_i}), y_2(t_{i,m_i}))\}$ denote the complete degradation observations of the ith sample. Both $Y_1(t)$ and $Y_2(t)$ are observed at the m_i observation time points. We further denote the incomplete degradation observations for the ith sample as

$$\mathbf{Y}_i^{1:h_i} = \{(y_1(t_{i,1}), y_2(t_{i,1})), \ldots, (y_1(t_{i,h_i}), y_2(t_{i,h_i})), (\cdot, y_2(t_{i,h_i+1})), \ldots, (\cdot, y_2(t_{i,m_i}))\} \quad (11)$$

where $(\cdot, y_2(t_{i,h_i+1}))$ indicates that, at the specific observation time point t_{i,h_i+1}, only $Y_2(t)$ is observed, whereas the observation of $Y_1(t)$ is missing. h_i is the number of observed degradation observations of $Y_1(t)$ for the ith sample.

B. Two-Stage Parameter Estimation Method

When the bivariate degradation process model presented in (7) is used for degradation modeling, we categorize the model parameters into three groups: $\boldsymbol{\theta}_{\mathrm{IG1}}$, $\boldsymbol{\theta}_{\mathrm{IG2}}$, and $\boldsymbol{\theta}_{\mathrm{C}}$. A two-stage parameter estimation method is introduced to facilitate the parameter estimation. The basic idea of this method originated from the work of Sari *et al.* [16] and Joe [35]. Specifically, the first stage involves the estimation of the model parameters for the marginal degradation processes, and the second stage involves the estimation of model parameters for the copula function based on the input synthesized from the first stage. By leveraging the flexibility of the Bayesian method [36] and the two-stage parameter estimation method [16], [35], a Bayesian version of the two-stage parameter estimation method is presented. A pictorial description of the two-stage parameter estimation method is depicted in Fig. 3.

As shown in Fig. 3, the first stage is to estimate the parameters of $Y_1(t)$ and $Y_2(t)$. This estimation is performed by utilizing the property of the proposed bivariate degradation process model. This property states that the marginal distributions of the degradation increments are $\mathrm{IG}(\Delta\Lambda_1(t_{i,j}), \lambda_1\Delta\Lambda_1(t_{i,j})^2|\boldsymbol{\theta}_{\mathrm{IG1}})$ and $\mathrm{IG}(\Delta\Lambda_2(t_{i,j}), \lambda_2\Delta\Lambda_2(t_{i,j})^2|\boldsymbol{\theta}_{\mathrm{IG2}})$, and the distribution of the CDFs of the degradation increments are uniform distributions as $F_1(\Delta y_{1ij}) \sim \mathrm{Uniform}(0,1)$ and $F_2(\Delta y_{2ij}) \sim \mathrm{Uniform}(0,1)$. Given the incomplete degradation observations $\mathbf{Y}_i^{1:h_i}, i = 1, \ldots, n$, the degradation increments for the two degradation processes can be obtained, and further used to estimate their respective model parameters. Using the Bayesian method, the specific procedure is mathematically formulated as

$$\begin{aligned} p(\boldsymbol{\theta}_{\mathrm{IG1}}|\Delta\mathbf{y}_1) &\propto \pi(\boldsymbol{\theta}_{\mathrm{IG1}}) \times \prod_{i=1}^{n}\prod_{j=1}^{h_i} f\left(\Delta y_{1ij}|\Delta\Lambda_1(t_{i,j}), \lambda_1\Delta\Lambda_1(t_{i,j})^2\right) \\ p(\boldsymbol{\theta}_{\mathrm{IG2}}|\Delta\mathbf{y}_2) &\propto \pi(\boldsymbol{\theta}_{\mathrm{IG2}}) \times \prod_{i=1}^{n}\prod_{j=1}^{m_i} f\left(\Delta y_{2ij}|\Delta\Lambda_2(t_{i,j}), \lambda_2\Delta\Lambda_2(t_{i,j})^2\right) \end{aligned} \quad (12)$$

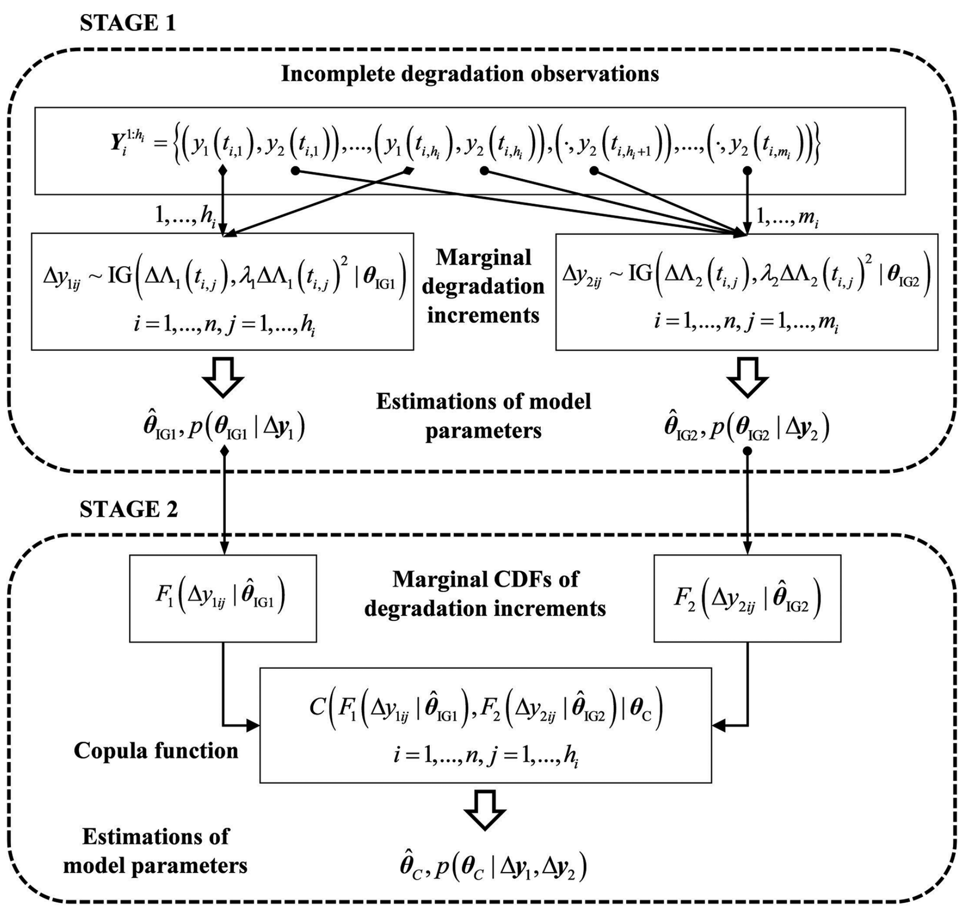

Fig. 3. Description of the two-stage estimation method.

where the function $f(\Delta y_{1ij}|\Delta\Lambda_1(t_{i,j}),\lambda_1\Delta\Lambda_1(t_{i,j})^2)$ is the PDF of the IG distribution given in (1) with specific parameters $a=\Delta\Lambda_1(t_{i,j})$ and $b=\lambda_1\Delta\Lambda_1(t_{i,j})^2$.

The Markov chain Monte Carlo method (MCMC) [37] is used to generate posterior samples from the posterior distributions presented in (12). The point estimations and interval estimations of the model parameters are obtained by statistically summarizing the generated posterior samples. For detailed information about the Bayesian estimation of model parameters of the IG degradation process, please refer to Peng *et al.* [28].

The second stage is to estimate the model parameters of the copula function. Based on the estimation of the model parameters for the degradation processes, the values of the CDFs of the degradation increments can be calculated and are $F_1(\Delta y_{1ij}|\hat{\boldsymbol{\theta}}_{\mathrm{IG1}})$ and $F_2(\Delta y_{2ij}|\hat{\boldsymbol{\theta}}_{\mathrm{IG2}})$. Each pair of the CDFs, $(F_1(\Delta y_{1ij}|\hat{\boldsymbol{\theta}}_{\mathrm{IG1}}),F_2(\Delta y_{2ij}|\hat{\boldsymbol{\theta}}_{\mathrm{IG2}}))$ is a sample from the copula function $C(F_1(\Delta y_{1ij}),F_2(\Delta y_{2ij})|\boldsymbol{\theta}_{\mathrm{C}})$. Therefore, the model parameters of the copula function can be estimated based on these samples. Using the Bayesian method, the estimation of $\boldsymbol{\theta}_{\mathrm{C}}$ is given as

$$\begin{aligned}
p(\boldsymbol{\theta}_{\mathrm{C}}|\Delta\mathbf{y}_1,\Delta\mathbf{y}_2)&\propto\pi(\boldsymbol{\theta}_{\mathrm{C}})\times\prod_{i=1}^{n}\prod_{j=1}^{h_i}c(\hat{u}_{ij},\hat{v}_{ij}|\boldsymbol{\theta}_{\mathrm{C}})\\
\hat{u}_{ij}&=F_1\left(\Delta y_{1ij}|\hat{\boldsymbol{\theta}}_{\mathrm{IG1}}\right),\hat{v}_{ij}=F_2\left(\Delta y_{2ij}|\hat{\boldsymbol{\theta}}_{\mathrm{IG2}}\right)\\
c(\hat{u}_{ij},\hat{v}_{ij}|\boldsymbol{\theta}_{\mathrm{C}})&=\frac{\partial^2 C(\hat{u}_{ij},\hat{v}_{ij}|\boldsymbol{\theta}_{\mathrm{C}})}{\partial\hat{u}_{ij}\partial\hat{v}_{ij}}
\end{aligned}\tag{13}$$

where $c(\hat{u}_{ij},\hat{v}_{ij}|\boldsymbol{\theta}_{\mathrm{C}})$ is the density of the bivariate copula function [18], [30].

The estimation of $\boldsymbol{\theta}_{\mathrm{C}}$ is based on the pairs of estimated CDFs of the degradation increments. Only the complete part of the degradation observations presented in (11) is used. The MCMC method is used to simulate posterior samples from the posterior distribution presented in (13). The point estimation and interval estimation are then summarized from the generated posterior samples.

C. Degradation Inference and RUL Prediction

When the model parameters are estimated, two different methods are introduced to individually implement the degradation inferences for missing observation points and for future observation points.

For the inference of missing degradation observations $(\cdot,y_2(t_{i,e_i})),e_i=h_i+1,\ldots,m_i$, a conditional copula based method is introduced. This method is based on the dependence structure of the bivariate degradation process model presented in (7). According to the properties of the copula function, if the degradation increments Δy_{2ie_i} are available based on the degradation observations, the following relationship is obtained:

$$\begin{aligned}
F(u_{ie_i}|v_{ie_i},\boldsymbol{\theta}_{\mathrm{C}})=C(u_{ie_i}|v_{ie_i},\boldsymbol{\theta}_{\mathrm{C}})&=\frac{\partial C(u_{ie_i},v_{ie_i}|\boldsymbol{\theta}_{\mathrm{C}})}{\partial v_{ie_i}}\\
u_{ie_i}&=F_1(\Delta y_{1ie_i}|\boldsymbol{\theta}_{\mathrm{IG1}})\\
v_{ie_i}&=F_2(\Delta y_{2ie_i}|\boldsymbol{\theta}_{\mathrm{IG2}})
\end{aligned}\tag{14}$$

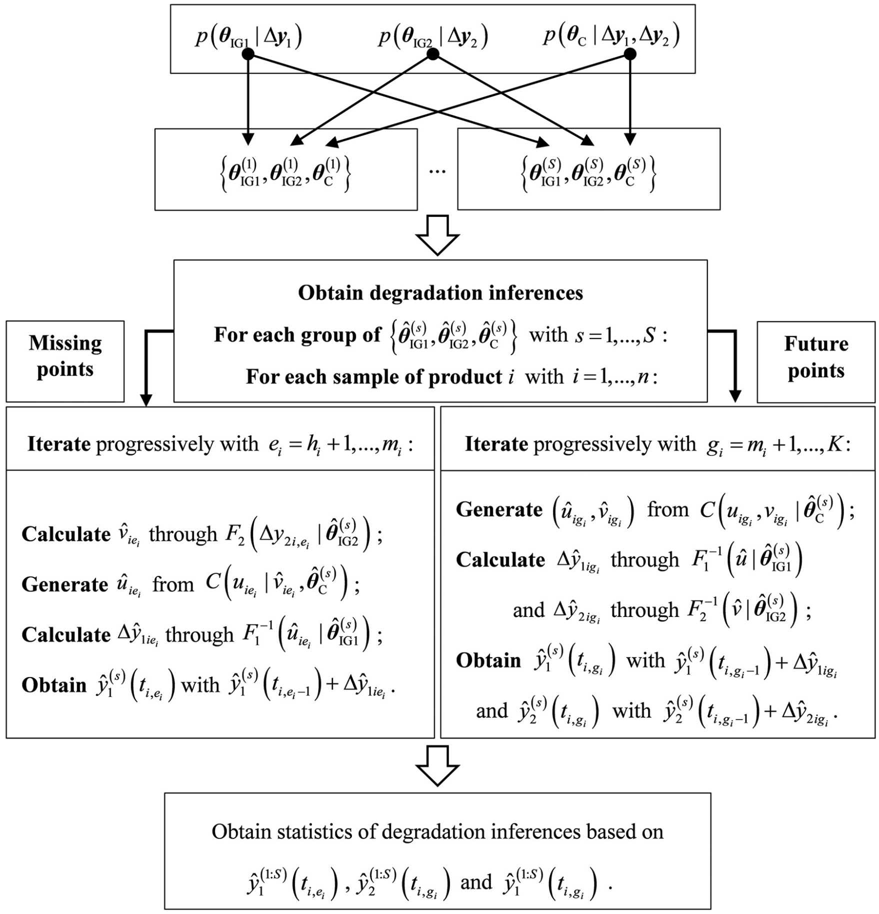

Fig. 4. Flowchart of methods for degradation inferences.

where $C(u_{ie_i}|v_{ie_i}, \boldsymbol{\theta}_{\mathrm{C}})$ is the conditional copula function given v_{ie_i} [18], [30].

By incorporating the estimations of $\boldsymbol{\theta}_{\mathrm{C}}$ and $\boldsymbol{\theta}_{\mathrm{IG2}}$ into (14), the conditional distribution of u_{ie_i} is obtained as

$$\begin{aligned}F\left(u_{ie_i}|\Delta\mathbf{y}_1,\Delta\mathbf{y}_2\right)=\int_{\boldsymbol{\theta}_{\mathrm{C}},\boldsymbol{\theta}_{\mathrm{IG2}}} & C\left(u_{ie_i}|v_{ie_i},\boldsymbol{\theta}_{\mathrm{C}}\right)\\ &\times p\left(\boldsymbol{\theta}_{\mathrm{C}}|\Delta\mathbf{y}_1,\Delta\mathbf{y}_2\right)\\ &\times p\left(\boldsymbol{\theta}_{\mathrm{IG2}}|\Delta\mathbf{y}_2\right)\mathrm{d}\boldsymbol{\theta}_{\mathrm{C}}\mathrm{d}\boldsymbol{\theta}_{\mathrm{IG2}}. \quad (15)\end{aligned}$$

Based on the simulation of u_{ie_i} through (15), the inferences of Δy_{1ie_i} are obtained according to the function relationship, $\Delta y_{1ie_i} = F_1^{-1}(u_{ie_i}|\boldsymbol{\theta}_{\mathrm{IG1}})$, where $F_1^{-1}(u_{ie_i}|\boldsymbol{\theta}_{\mathrm{IG1}})$ is the inverse of the IG CDF given in (2). The degradation inference of $y_1(t_{ie_i})$ is then obtained as

$$\begin{aligned}y_1\left(t_{i,e_i}\right)&=y_1\left(t_{i,e_i-1}\right)+\Delta y_{1ie_i},\\ \Delta y_{1ie_i}&=\int_{\boldsymbol{\theta}_{\mathrm{IG1}}}F_1^{-1}\left(u_{ie_i}|\boldsymbol{\theta}_{\mathrm{IG1}}\right)p\left(\boldsymbol{\theta}_{\mathrm{IG1}}|\Delta\mathbf{y}_1\right)\mathrm{d}\boldsymbol{\theta}_{\mathrm{IG1}}. \quad (16)\end{aligned}$$

The calculations of (15) and (16) are carried out using a simulation based integration method. The procedure for this simulation based integration method is presented in Fig. 4. The calculations are based on the posterior samples of model parameters, $\{\hat{\boldsymbol{\theta}}_{\mathrm{IG1}}^{(s)}, \hat{\boldsymbol{\theta}}_{\mathrm{IG2}}^{(s)}, \hat{\boldsymbol{\theta}}_{\mathrm{C}}^{(s)}\}$ with $s = 1, \ldots, S$, which are generated from the posterior distributions in (12) and (13). For each group of posterior samples, the CDF of Δy_{2ie_i} is calculated. The CDF of Δy_{1ie_i} is obtained through the conditional copula function, as presented in (14). The degradation increment $\Delta\hat{y}_{1ie_i}$ is then obtained by calculating the inverse IG CDF, $F_1^{-1}(\hat{u}_{ie_i}|\hat{\boldsymbol{\theta}}_{\mathrm{IG1}}^{(s)})$. A sample of degradation inference for the missing observation is obtained as $\hat{y}_1^{(s)}(t_{i,e_i})$. By repeating these procedures S times, a group of degradation inferences are obtained as $\hat{y}_1^{(1:S)}(t_{i,e_i})$. Statistical summarization can then be drawn based on these samples, which includes the mean, variance, empirical PDF, empirical CDF, and so on.

For degradation inferences of future observation points $(y_1(t_{i,g_i}), y_2(t_{i,g_i})), g_i = m_i + 1, \ldots, K$, a copula function-based method is introduced, where K is the number of future observation points interested. Because neither the degradation observation of $Y_1(t)$ nor $Y_2(t)$ is available, degradation inferences of $(y_1(t_{i,g_i}), y_2(t_{i,g_i})), g_i = m_i + 1, \ldots, K$ are given as

$$\begin{aligned}y_1\left(t_{i,g_i}\right)&=y_1\left(t_{i,g_i-1}\right)+\Delta y_{1ig_i}, y_2\left(t_{i,g_i}\right)\\ &=y_2\left(t_{i,g_i-1}\right)+\Delta y_{2ig_i}\\ \Delta y_{1ig_i}&=\int_{\boldsymbol{\theta}_{\mathrm{IG1}}}F_1^{-1}\left(u_{ig_i}|\boldsymbol{\theta}_{\mathrm{IG1}}\right)p\left(\boldsymbol{\theta}_{\mathrm{IG1}}|\Delta\mathbf{y}_1\right)\mathrm{d}\boldsymbol{\theta}_{\mathrm{IG1}}\\ \Delta y_{2ig_i}&=\int_{\boldsymbol{\theta}_{\mathrm{IG2}}}F_2^{-1}\left(v_{ig_i}|\boldsymbol{\theta}_{\mathrm{IG2}}\right)p\left(\boldsymbol{\theta}_{\mathrm{IG2}}|\Delta\mathbf{y}_2\right)\mathrm{d}\boldsymbol{\theta}_{\mathrm{IG2}},\\ F_{UV}\left(u_{ig_i},v_{ig_i}\right)&=\int_{\boldsymbol{\theta}_{\mathrm{C}}}C\left(u_{ig_i},v_{ig_i}|\boldsymbol{\theta}_{\mathrm{C}}\right)p\\ &\quad\times\left(\boldsymbol{\theta}_{\mathrm{C}}|\Delta\mathbf{y}_1,\Delta\mathbf{y}_2\right)\mathrm{d}\boldsymbol{\theta}_{\mathrm{C}}. \quad (17)\end{aligned}$$

The calculation of (17) is implemented through a simulation-based integration method. A pictorial description of this method is presented in Fig. 4. Similar to the method for degradation inference of missing observations, the calculations are based on the posterior samples of model parameters $\{\hat{\boldsymbol{\theta}}_{\mathrm{IG1}}^{(s)}, \hat{\boldsymbol{\theta}}_{\mathrm{IG2}}^{(s)}, \hat{\boldsymbol{\theta}}_{\mathrm{C}}^{(s)}\}$. For each group of posterior samples, a pair of $(\hat{u}_{ig_i}, \hat{v}_{ig_i})$ is generated from $C(u_{ig_i}, v_{ig_i}|\hat{\boldsymbol{\theta}}_{\mathrm{C}}^{(s)})$. $\Delta\hat{y}_{1ig_i}$ and $\Delta\hat{y}_{2ig_i}$ are obtained by calculating the inverse IG CDFs, $F^{-1}(\hat{u}_{ig_i}|\hat{\boldsymbol{\theta}}_{\mathrm{IG1}}^{(s)})$ and $F^{-1}(\hat{v}_{ig_i}|\hat{\boldsymbol{\theta}}_{\mathrm{IG2}}^{(s)})$. A pair of degradation inferences for the future observation is then obtained as $(\hat{y}_{1ig_i}, \hat{y}_{2ig_i})$. By repeating these procedures S times, a group of degradation inferences are obtained as $(\hat{y}_{1ig_i}, \hat{y}_{2ig_i})^{(1:S)}$. By summarizing these samples of degradation inferences, the mean, variance, and interval estimation of the degradation observations at future time points are obtained.

The estimation of reliability and the prediction of RUL are dependent on the degradation inferences for the future points, which have been presented in (17). Given the degradation threshold, the reliability estimation and the RUL predication are mathematically described as

$$\begin{aligned} y_1\left(t_{i,g_i}\right) &= y_1\left(t_{i,g_i-1}\right) + \Delta y_{1ig_i}, y_2\left(t_{i,g_i}\right) \\ &= y_2\left(t_{i,g_i-1}\right) + \Delta y_{2ig_i} \\ R_i\left(t_{i,g_i}\right) &= \Pr\left\{Y_1\left(t_{i,g_i}\right) \le D_1, Y_2\left(t_{i,g_i}\right) \le D_2\right\} \\ T_i^{\mathrm{RUL}} &= \inf\{L : Y_1(t_{i,h_i} + L) > D_1 \end{aligned}$$

or

$$\begin{aligned} &Y_2(t_{i,h_i} + L) > D_2\} \\ &\qquad L = t_{i,g_i} - t_{i,h_i}, g_i = h_i, \ldots, K \to +\infty \end{aligned} \tag{18}$$

where $y_1(t_{i,g_i})$ and $y_2(t_{i,g_i})$ are the degradation inferences presented in (17) for future time points t_{i,g_i}, and the range of g_i is extended to infinity to cover the possible failure points in the future.

Because there are no analytical forms for the reliability function and RUL distribution, a simulation based method is used for the estimation of reliability and the prediction of RUL. A pictorial description of the simulation based method for RUL estimation is presented in Fig. 5.

This method is based on the posterior samples of the model parameters generated from the posterior distributions given in (12) and (13). For each group of posterior samples, unit-specific RULs are predicted for the products, where different products may have different RULs according to their respective observations at the latest observing time t_{i,h_i}. The unit-specific RULs are obtained by progressively generating degradation inferences until the time point at which one of the degradation inferences reaches its failure threshold. This specific point is the failure time of the corresponding product, and the time interval between this point and t_{i,g_i} is the RUL of the product. By obtaining the failure times for all the groups of model parameters, the statistics of the reliability function and the RUL can then be obtained.

IV. Simulation Study

Here, a simulation study is presented to investigate the effectiveness of the proposed method for dependence modeling, degradation inference, and RUL prediction under different dependence situations. Two IG processes and a Gaussian copula function are used to generate degradation observations. Three groups of degradation observations are generated and presented in Fig. 1. These observations are generated using the following parameter cases: $\mu_1 = 3$, $\lambda_1 = 0.35$, $\mu_2 = 1$, and $\lambda_2 = 0.35$ for the marginal degradation processes; and different linear coefficients, $\alpha = 0, 0.5, 0.99$ for the copula function to reflect the different degrees of dependence between these two marginal degradation processes. The incomplete degradation observations are then generated by artificially removing the degradation observations at the missing observation points, where real values are reserved for the validation of the degradation inferences. A group of incomplete degradation observations with $\alpha = 0.99$ are shown in Fig. 2. These artificial missing observations include the following points: $y_1(t_{1,11})$, $y_1(t_{3,10})$, $y_1(t_{3,11})$, and $y_1(t_{5,11})$.

Because the proposed model can model different dependence situations through the copula function, the proposed method is used directly on the generated data without checking the strength of dependence between the degradation processes. Following the procedures of the two-stage Bayesian estimation method, the first stage is implemented, and the estimations of the model parameters for the marginal degradation processes are obtained. A pictorial description of the relative errors of the estimated model parameters μ_1, λ_1, μ_2, and $\lambda_2$is presented in Fig. 6. This figure shows the boxplots of the relative errors with mean values displayed as dotted circles in the figure, and the 95% confidence intervals are presented as vertical columns. These relative errors are the ratios of the errors of the estimated values to the real values of the model parameters. A high-precision of model parameter estimation is demonstrated for the proposed method through the low relative errors shown in the figure.

Based on the estimations of the model parameters, the CDFs of the degradation increments are obtained. Scatter plots of the CDFs of the degradation increments are depicted in Fig. 7. These scatter plots are qualitative descriptions of the dependence between the two marginal degradation processes.

The second-stage of the two-stage method is then implemented based on the CDFs of the degradation increments. The parameter of the copula function is then estimated and is also presented in Fig. 6 as α. The accuracy of the estimation results is demonstrated by the low relative errors of α.

Based on the posterior samples of the model parameters, the simulation based degradation inferences for the missing observation points are obtained, and are presented in Fig. 8.

A high precision of degradation inferences for the missing degradation points is demonstrated under the proposed method, especially for the situation with high degree of dependence between the degradation processes. This high precision is due to the modeling of dependence through the copula function. A connection is then formulated between the two degradation processes, through which the degradation inferences of one degradation process can rely on the degradation observations of the other degradation process. To further demonstrate the effectiveness of dependence modeling, the degradation inferences for

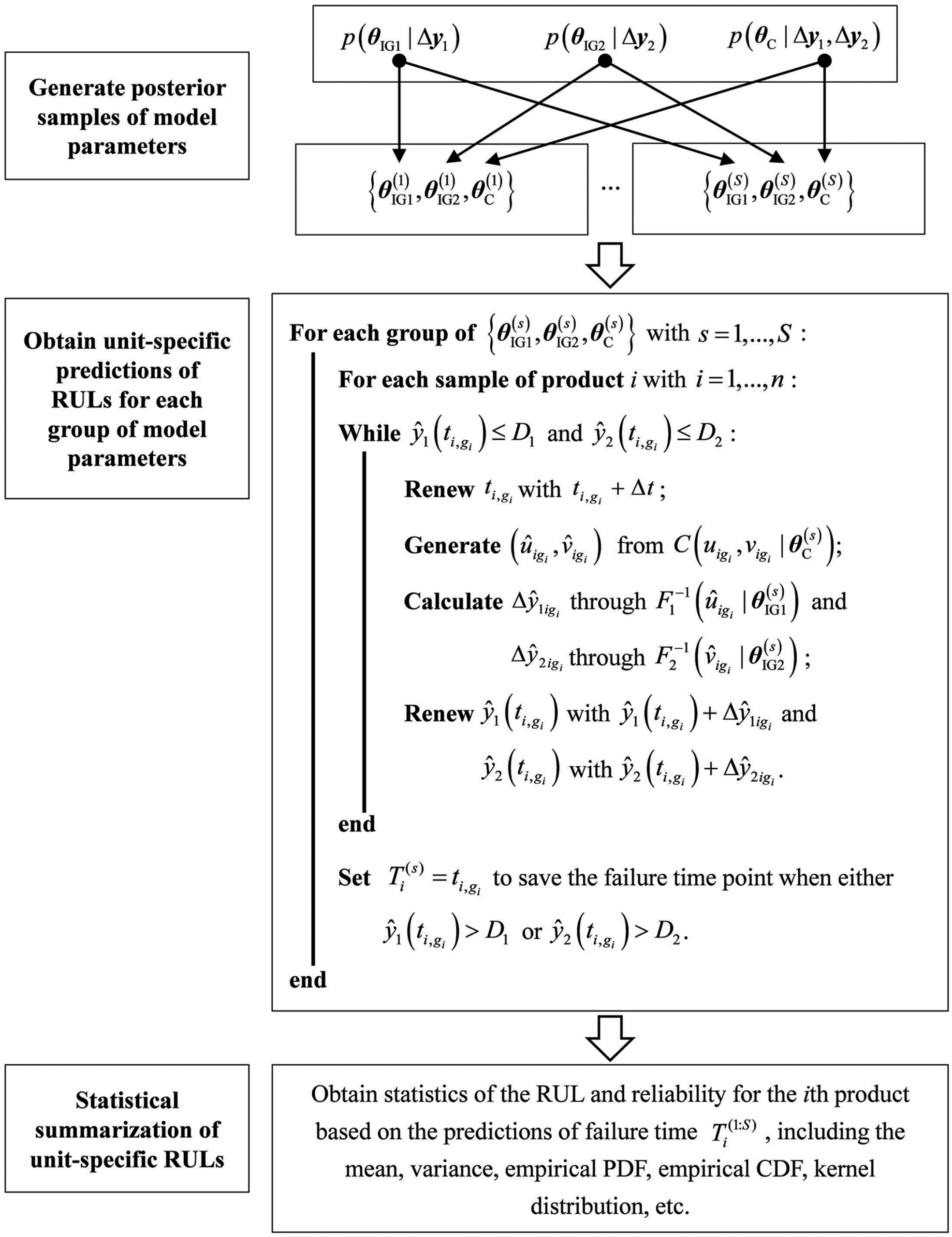

Fig. 5. Pictorial description of the simulation based RUL prediction.

missing degradation observations without considering this dependence are obtained, and are presented in Fig. 9. These inferences are obtained using the same estimations of the model parameters presented above, except for $\alpha = 0$, to make the two degradation processes s-independent.

When the dependence between the degradation processes increases, discrepancy between the real values and the inferred values is observed under the s-independent degradation process model. This discrepancy indicates that ignorance of the dependence among degradation processes can lead to poor inferences of the missing degradation observations when there is a strong dependence between the two degradation processes. This ignorance can further lead to imprecise estimations of RULs, which rely heavily on the degradation inferences as presented in (18). To demonstrate this point, the relative errors of the RUL predictions between the proposed model and the s-independent model are compared. The comparison results are shown in Fig. 10. Compared with the independent model, higher precision of RUL predictions are achieved by the proposed model under the simulation case with $D_1 = 300$, $D_2 = 100$, and $\alpha = 0.99$, where a strong dependence between these two degradation processes exists.

To further demonstrate the dependence modeling in the proposed model, comparisons of the degradation inferences for the missing point $y_1(t_{3,11})$ generated by the proposed method and the s-independent model under the case with $\alpha = 0.99$ are presented. Given the incomplete degradation observation, $(\cdot, y_2(t_{3,11}))$, where $y_2(t_{3,11})$ is available and $y_1(t_{3,11})$ is missing, the degradation inferences of $y_1(t_{3,11})$ generated by different models are given in Fig. 11.

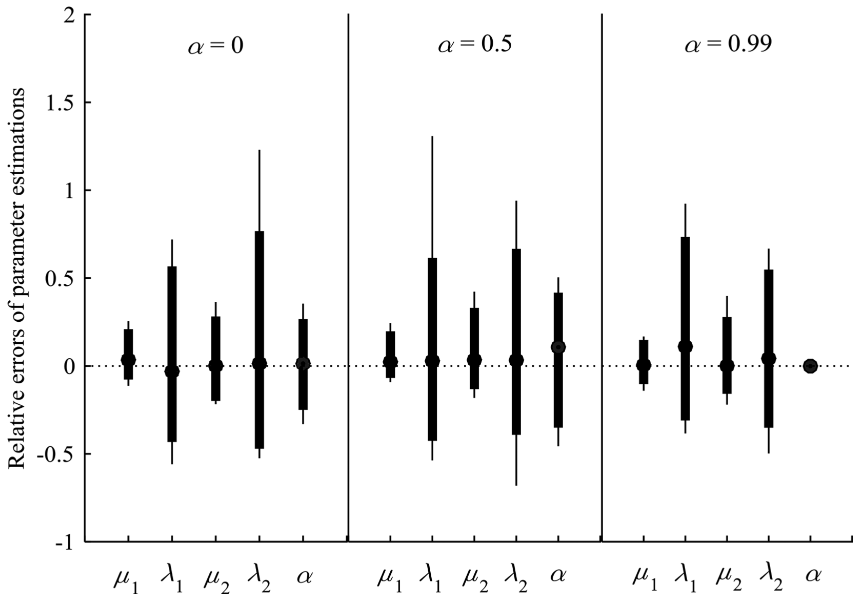

Fig. 6. Relative errors of the parameter estimations obtained by the two-stage method.

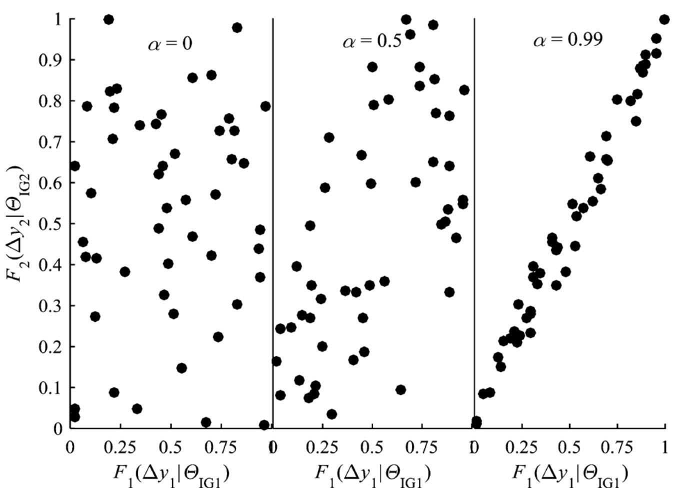

Fig. 7. Scatter plots of the CDFs of the degradation increments.

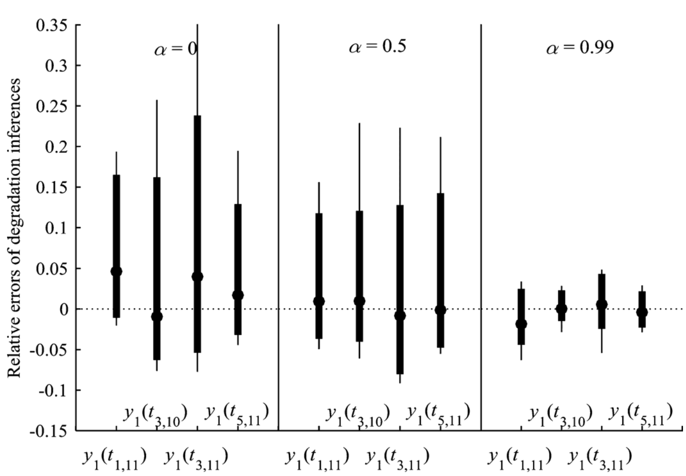

Fig. 8. Relative errors of the degradation inferences for the missing observations by the proposed model.

A distribution of $y_1(t_{3,11})$ with smaller variance is obtained under the proposed model compared with the s-independent model. According to (16), the distribution of $y_1(t_{3,11})$ is obtained based on samples of $u(t_{3,11})$ through the inverse IG CDF, $F_1^{-1}(u(t_{3,11})|\boldsymbol{\theta}_{\mathrm{IG1}})$. In the proposed model, a connection constructed by the copula function between $u(t_{3,11})$ and $v(t_{3,11})$ can help to improve the inference precision of $y_1(t_{3,11})$. Further examination of the samples of $(u(t_{3,11}), v(t_{3,11}))$ under the proposed model and the s-independent model provides a more detailed interpretation. Fig. 12 presents a comparison of the samples of $(u(t_{3,11}), v(t_{3,11}))$, and the corresponding distributions between the proposed model and the s-independent model.

For the s-independent model, the samples of $u(t_{3,11})$ have no relationship with the samples of $v(t_{3,11})$. The $v(t_{3,11})$ exert no inference on the sampling of $u(t_{3,11})$. A uniform distribution is obtained for the distribution of $u(t_{3,11})$ under the s-independent model. However, for the proposed model, the samples of $u(t_{3,11})$ can be obtained through the relationship $C(u(t_{3,11})|v(t_{3,11}), \boldsymbol{\theta}_{\mathrm{C}})$ given in (14), when the samples of $v(t_{3,11})$ are available. A distribution with a smaller diffusion of $u(t_{3,11})$ is obtained under the proposed model, which in turn leads to the converged distribution of $y_1(t_{3,11})$ presented in Fig. 11. The same effects are observed for the degradation inferences of other missing observation points. Accordingly, the effectiveness of the proposed method for the degradation inferences of missing degradation observations is verified.

V. Illustrative Example

Here, an application of the proposed method for the bivariate degradation analysis of heavy machine tools subject to incomplete measurements is presented. To maintain the high availability and high efficiency of heavy machine tools, PM and SHM are implemented. The positioning accuracy and output power are two indispensable performance indicators for the PM and degradation analysis. Measurements of the positioning accuracy are performed by programmed procedures, where groups of continually updating observations are observed. However, measurements of the output power are missing at some

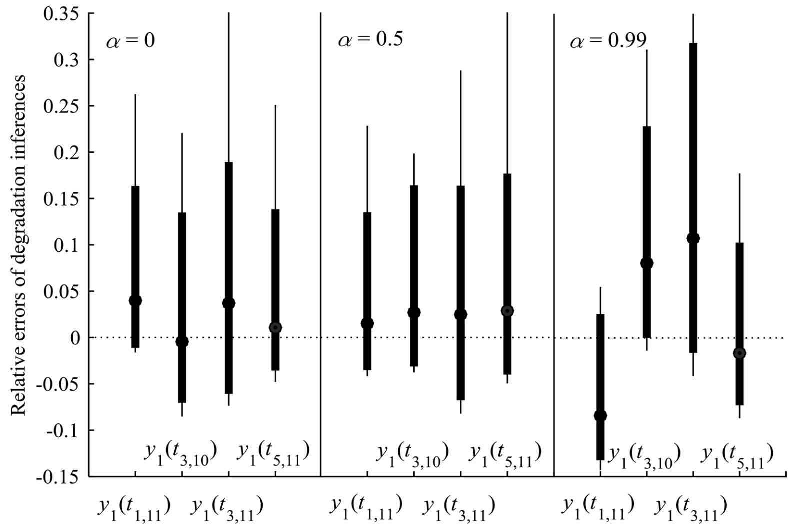

Fig. 9. Relative errors of the degradation inferences for the missing observations by the s-independent model.

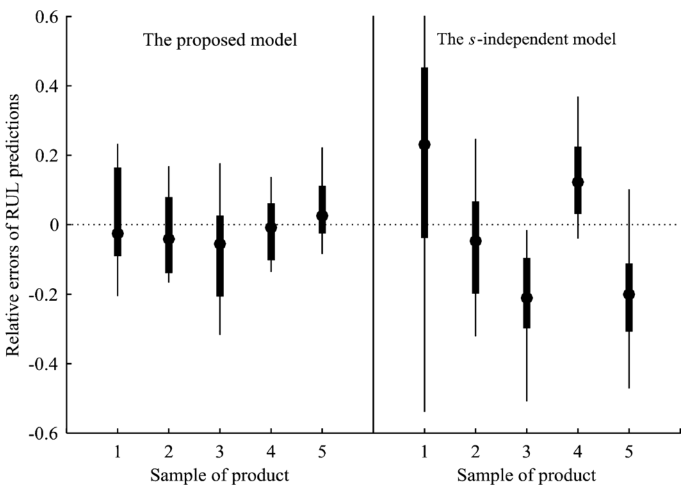

Fig. 10. Comparison of the relative errors of the RUL predictions between the proposed model and the s-independent model.

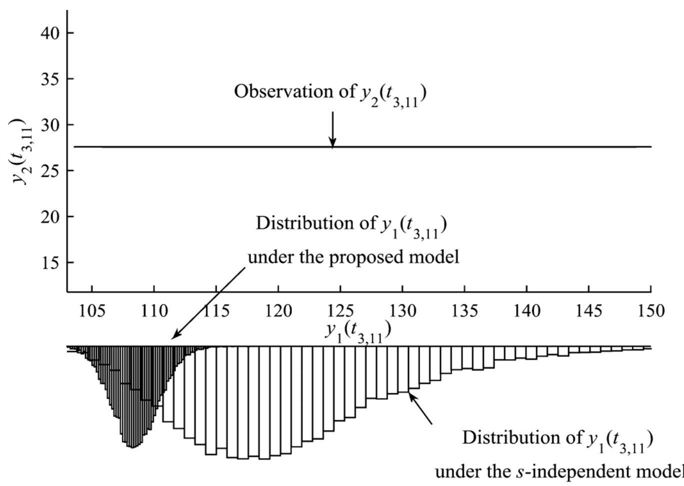

Fig. 11. Comparison of the distributions of $y_1(t_{3,11})$ between the proposed model and the s-independent model.

observation points due to interruptions caused by changing of the manufacturing schedules or unavailability of the measuring system operators, leading to incomplete observations for the degradation analysis. In addition, historical information and experts' experience have indicated that these two performance indicators are correlated. As a result, the degradation processes of the positioning accuracy and output power are characterized as bivariate s-dependent degradation processes with incomplete degradation observations. A suitable degradation modeling and a precise degradation analysis of these two degradation processes are critical for the operation and management of heavy machine tools.

A. Incomplete Degradation Observations

The degradation observations of the positioning accuracy and output power of three heavy machine tools are given in Table I and are presented in Fig. 13. The degradation measurement of the positioning accuracy $y_1(t_{i,j})$ is available at each observation point $t_{i,j}$. However, the degradation measurements of the output power $y_2(t_{i,j})$ are missing at some observation points, which are indicated in Table I. According to the performance requirements of the heavy machine tools, the thresholds for the degradations of the positioning accuracy and output power are $D_1 = 35$ and $D_2 = 120$. Due to proprietary issues, the units of the performance indicators are omitted, and the degradation observations are modified to some degree. Largely, however, the characteristics of bivariate s-dependent degradation processes with incomplete degradation observations are reserved for demonstration of the proposed method.

B. Degradation Modeling and Parameter Estimation

The multivariate IG process model introduced above is used to model the degradation observations presented in Table I. This choice is due to the fact that the degradation observations are characterized as monotone increasing degradation processes with either non-linear or linear degradation paths, as presented in Fig. 13. In addition, some dependence among these two degradation processes has been suggested by historical information and experts' experience. The IG process model can describe these characteristics by incorporating different forms of degradation mean functions. In detail, a linear degradation

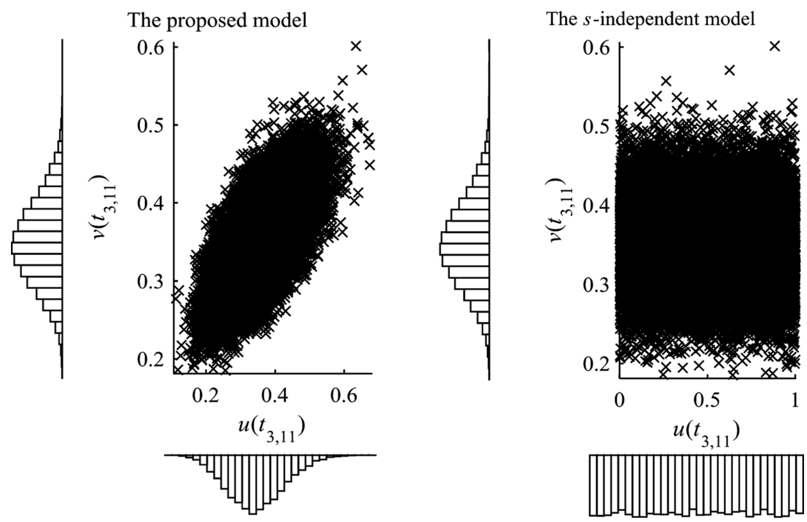

Fig. 12. Comparison of samples and the distribution of $(u(t_{3,11}), v(t_{3,11}))$ between the proposed model and the s-independent model.

TABLE I
DEGRADATION MEASUREMENTS OF THE POSITIONING ACCURACY $y_1(t_{i,j})$ AND OUTPUT POWER$y_2(t_{i,j})$ OF THREE HEAVY MACHINE TOOLS AT OBSERVATION TIME POINTS $t_{i,j}$

	$t_{1,j}$	4	6	10	11	12	13	16	17	19	22	29
Sample 1	$y_1(t_{1,j})$	2.22	3.050	8.89	9.23	9.660	9.73	11.31	11.57	12.47	13.62	18.32
	$y_2(t_{1,j})$	1.08	2.20	11.83	13.29	14.96	15.63	23.00	24.97	31.03	40.79	74.99
	$t_{2,j}$	1	3	4	5	9	13	14	17	22	25	26
Sample 2	$y_1(t_{2,j})$	0.41	1.29	1.80	3.79	8.49	11.21	13.24	18.52	22.79	27.64	28.41
	$y_2(t_{2,j})$	0.02	0.38	0.83	2.83	10.10	18.84	23.49	37.24	•	•	•
	$t_{3,j}$	2	7	8	11	12	14	17	21	23	24	29
Sample 3	$y_1(t_{3,j})$	1.62	5.53	5.70	13.55	13.66	15.80	17.67	19.37	23.06	23.26	25.32
	$y_2(t_{3,j})$	0.29	4.95	5.60	16.28	17.09	23.36	32.06	43.75	56.47	59.61	•

where '•' indicates that there is a missing of degradation observation.

mean function is chosen for the degradation process of the positioning accuracy, and a power-law degradation mean function is chosen for the degradation process of the output power. The bivariate IG process model is then given as

$$\begin{aligned} y_1(t_{i,j}) &\sim \mathrm{IG}\left(\Lambda_1(t_{i,j}), \lambda_1\Lambda_1(t_{i,j})^2\right), \Lambda_1(t_{i,j}) = \mu_1 t_{i,j} \\ y_2(t_{i,j}) &\sim \mathrm{IG}\left(\Lambda_2(t_{i,j}), \lambda_2\Lambda_2(t_{i,j})^2\right), \Lambda_2(t_{i,j}) = \mu_2 t_{i,j}^q \\ F(\Delta y_{1ij}, \Delta y_{2ij}) &= C(F_1(\Delta y_{1ij}), F_2(\Delta y_{2ij})) \end{aligned} \tag{19}$$

where the copula function is not specified because the characteristic of dependence is not identified clearly at this moment.

According to the two-stage parameter estimation method presented in Section III, the model parameters, $\boldsymbol{\theta}_{\mathrm{IG1}} = \{\mu_1, \lambda_1\}$ and $\boldsymbol{\theta}_{\mathrm{IG2}} = \{\mu_2, \lambda_2, q\}$, for $Y_1(t)$ and $Y_2(t)$ are estimated using the Bayesian method based on the degradation observations presented in Table I. Noninformative priors of the model parameters are used in the Bayesian estimation, which are given as uniform distributions within large intervals. Based on (12), the posterior distributions of model parameters are

$$\begin{aligned} p\left(\mu_1, \lambda_1 | \Delta\mathbf{y}_1\right) &\propto \prod_{i=1}^{3}\prod_{j=1}^{11}\sqrt{\lambda_1\mu_1^2} \\ &\quad\times \exp\left(-\frac{\lambda_1\left(\Delta y_{1ij} - \mu_1\Delta t_{i,j}\right)^2}{2\Delta y_{1ij}}\right) \\ p\left(\mu_2, \lambda_2, q | \Delta\mathbf{y}_2\right) &\propto \prod_{i=1}^{3}\prod_{j=1}^{h_i}\sqrt{\lambda_2\mu_2^2\left(t_{i,j}^q - t_{i,j-1}^q\right)^2} \\ &\quad\times \exp\left(-\frac{\lambda_2\left(\Delta y_{2ij} - \mu_2\left(t_{i,j}^q - t_{i,j-1}^q\right)\right)^2}{2\Delta y_{2ij}}\right) \\ \Delta y_{1ij} &= y_1\left(t_{i,j}\right) - y_1\left(t_{i,j-1}\right) \\ \Delta y_{2ij} &= y_2\left(t_{i,j}\right) - y_2\left(t_{i,j-1}\right), \Delta t_{i,j} = t_{i,j} - t_{i,j-1} \end{aligned} \tag{20}$$

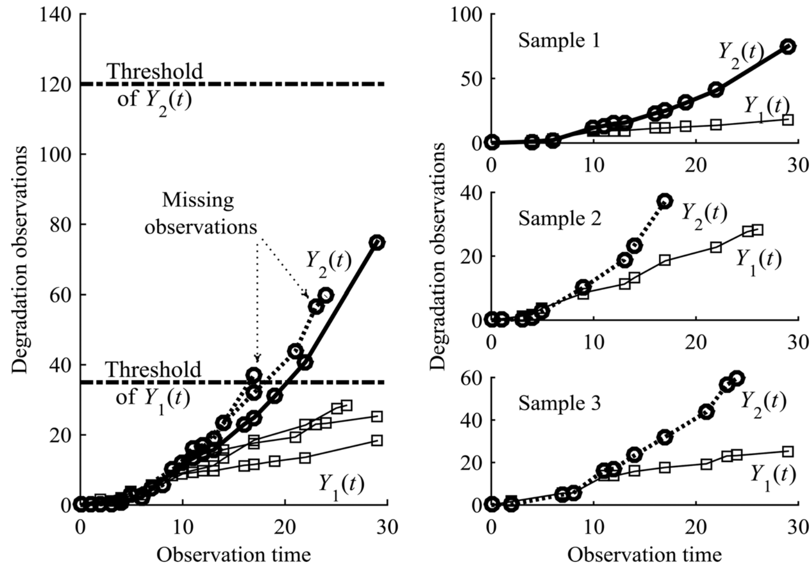

Fig. 13. Degradation observations of the positioning accuracy $Y_1(t)$ and output power $Y_2(t)$ of three heavy machine tools.

TABLE II
ESTIMATIONS OF THE MODEL PARAMETERS AND THE PRIORS USED IN THE BAYESIAN ESTIMATION

Parameter	Posterior		Posterior percentiles		Prior distribution
	Mean	**SD**	**2.5%**	**97.5%**	
μ_1	0.8754	0.1322	0.6752	1.1720	Uniform(0,100)
λ_1	0.8050	0.2858	0.3260	1.4350	Uniform(0,100)
μ_2	0.1621	0.0506	0.0861	0.2808	Uniform(0,100)
λ_2	1.1200	0.3551	0.5212	1.9130	Uniform(0,100)
q	1.8670	0.0910	1.6900	2.0450	Uniform(0,100)

where $y_1(t_{i,0}) = y_2(t_{i,0}) = 0$ and h_i is the number of degradation observations of the output power for the ith heavy machine tool, as contained in $\mathbf{h} = \{11, 8, 10\}$.

Posterior samples of the model parameters are generated from the joint posterior distribution presented in (20) using the MCMC method. Statistical summarizations of these posterior samples are presented in Table II; the prior distributions used in the Bayesian estimation are also provided.

Based on the estimations of the model parameters, the CDFs of the degradation increments, $F_1(\Delta y_{1ij})$ and $F_2(\Delta y_{2ij})$, are obtained. According to the bivariate degradation model presented in (19), a pictorial description of the relationship between the degradation increments, $(F_1(\Delta y_{1il}), F_2(\Delta y_{2il}))$ with $i = 1, 2, 3$ and $l = 1, \ldots, h_i$ is presented in Fig. 14.

A strong dependence is observed from Fig. 14. The Gaussian copula function is chosen to characterize the apparent lower-lower and upper-upper tail dependence depicted

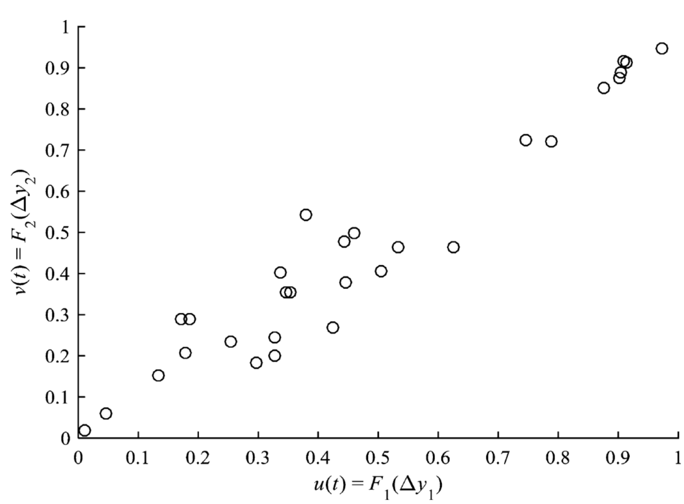

Fig. 14. Scatter plot of the CDFs of the degradation increments of the positioning accuracy and output power.

TABLE III
ESTIMATIONS OF THE MODEL PARAMETER α

Parameter	Posterior		Posterior percentiles	
	Mean	SD	2.5%	97.5%
α	0.9678	0.0096	0.9446	0.9812

in Fig. 14. Accordingly, the copula function of the bivariate degradation process model presented in (19) is specified as

$$\begin{aligned} F(\Delta y_{1ij}, \Delta y_{2ij}) &= C(F_1(\Delta y_{1ij}), F_2(\Delta y_{2ij})) \\ &= \int_{-\infty}^{\Phi^{-1}(u(t_{i,j}))} \int_{-\infty}^{\Phi^{-1}(v(t_{i,j}))} \frac{1}{2\pi\sqrt{1-\alpha^2}} \\ &\quad \times \exp\left(-\frac{x^2 - 2\alpha xy - y^2}{2(1-\alpha^2)}\right) \mathrm{d}x\mathrm{d}y \\ u(t_{i,j}) &= F_1(\Delta y_{1ij}), v(t_{i,j}) = F_2(\Delta y_{2ij}). \quad (21) \end{aligned}$$

Following the two-stage parameter estimation method, the model parameter $\boldsymbol{\theta}_{\mathrm{C}} = \{\alpha\}$ is estimated based on the CDFs of the degradation increments presented in Fig. 14. A uniform distribution, $\mathrm{Uniform}(-1, 1)$, within the bound of α is used as the non-informative prior distribution. The posterior distribution of α is then given as

$$\begin{aligned} p(\alpha|\Delta\mathbf{y}_1, \Delta\mathbf{y}_2) &\propto \prod_{i=1}^{3}\prod_{j=1}^{h_i} \frac{1}{\sqrt{1-\alpha^2}} \exp\left(-\frac{a^2 - 2\alpha ab - b^2}{2(1-\alpha^2)}\right) \\ a &= \Phi^{-1}(F_1(\Delta y_{1ij})) \\ b &= \Phi^{-1}(F_2(\Delta y_{2ij})), \ \mathbf{h} = \{11, 8, 10\}. \quad (22) \end{aligned}$$

Similarly, the MCMC method is used to generate posterior samples of α from the posterior distribution. Statistical summarizations of the posterior samples are presented in Table III.

C. Degradation Inference and RUL Prediction

Based on the posterior samples of $\boldsymbol{\theta}_{\mathrm{IG1}} = \{\mu_1, \lambda_1\}$, $\boldsymbol{\theta}_{\mathrm{IG2}} = \{\mu_2, \lambda_2, q\}$, and $\boldsymbol{\theta}_{\mathrm{C}} = \{\alpha\}$, degradation inferences of the missing observations are obtained following the procedure described in Section III. In detail, the degradation increments of the positioning accuracy, including $\Delta y_1(t_{2,9})$, $\Delta y_1(t_{2,10})$, $\Delta y_1(t_{2,11})$, and $\Delta y_1(t_{3,11})$, are used to infer their counterpart degradation increments of the output power through the Gaussian copula function. Given the CDFs of the degradation increments of the positioning accuracy, the corresponding CDFs of the degradation increments of the output power are obtained, which include $F_2(\Delta y_2(t_{2,9}))$, $F_2(\Delta y_2(t_{2,10}))$, $F_2(\Delta y_2(t_{2,11}))$, and $F_2(\Delta y_2(t_{3,11}))$. A pictorial description of the inferred CDFs of the degradation increments of output power, and the observed CDFs of the degradation increments of positioning accuracy is presented in Fig. 15.

Utilizing the inferred CDFs of the degradation increments of the output power, the degradation inferences of the missing degradation observations are obtained through (16). Because the degradation inferences are obtained based on posterior samples of the model parameters, groups of posterior inferences of missing degradation observations are obtained. Summarizations and boxplots of these posterior inferences are given in Table IV and Fig. 16.

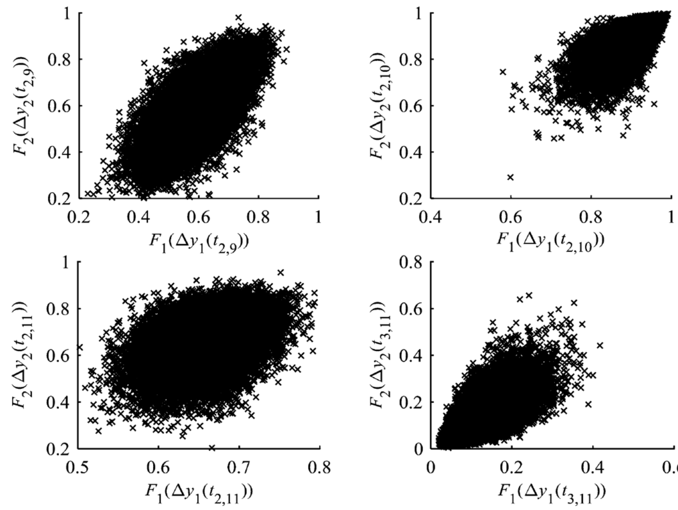

Fig. 15. Scatter plot of the CDFs of the degradation increments of the positioning accuracy and output power.

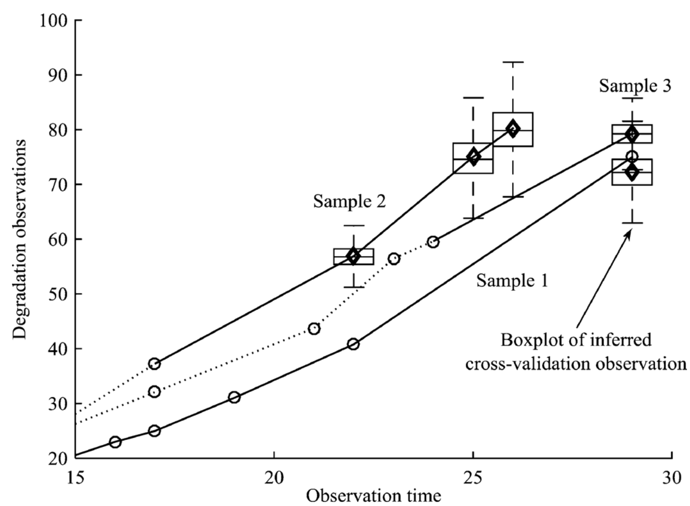

Fig. 16. Boxplot of the degradation inferences for the missing observations and the cross-validation observation.

TABLE IV
INFERENCES OF THE MISSING DEGRADATION OBSERVATIONS AND THE CROSS-VALIDATION OBSERVATION OF THE OUTPUT POWER

Parameter	Posterior		Posterior percentiles	
	Mean	SD	2.5%	97.5%
$y_2(t_{1,11})$	72.3070	3.5640	65.6415	79.7541
$y_2(t_{2,9})$	56.2580	4.6402	48.7133	67.0282
$y_2(t_{2,10})$	69.6644	6.3348	59.0339	83.9852
$y_2(t_{2,11})$	74.4539	6.9049	62.8281	90.0355
$y_3(t_{2,11})$	84.4378	5.5315	75.1971	96.8852

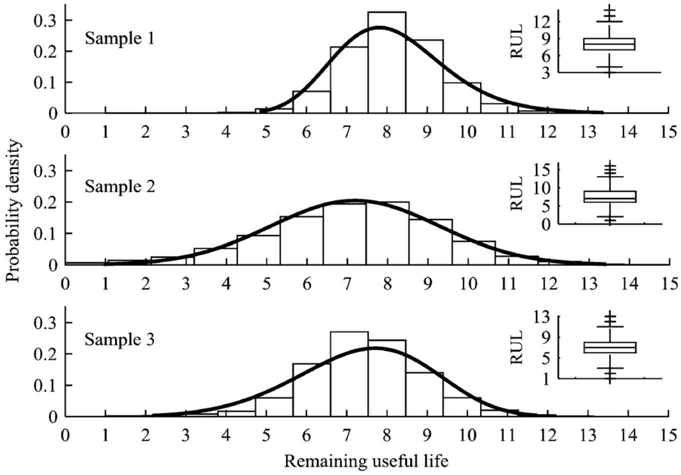

Fig. 17. Probability densities and boxplot of the RULs for the heavy machine tools.

To validate the capability of degradation inference, the observed degradation observation, $y_2(t_{1,11})$ is used as the leave-one out cross-validation point. The observed value of $y_2(t_{1,11})$ is 74.99. The inference of $y_2(t_{1,11})$ is summarized in Table IV. A relative error of 3.5774% is observed for the cross-validation degradation observation, which indicates that the precision of the degradation inferences for the missing observations is acceptable.

The posterior samples of the model parameters, and the degradation inferences of the missing degradation observations are further used to infer the degradations at future observation points. Following the procedure presented in Section III, degradation increments of the positioning accuracy and output power are generated through the Gaussian copula function and the marginal IG distributions. By comparing the degradation inferences for future observation points with the degradation thresholds of the two degradation processes, the RULs of the heavy machine tools are obtained, and are presented in Fig. 17. Operation and management of the heavy machine tools can then be performed based on the inferences of the missing output power and the predicted RULs of the heavy machine tools.

VI. Conclusion

This paper presents a coherent bivariate degradation analysis with incomplete degradation observations. A new type of bivariate degradation process model is introduced based on inverse Gaussian processes and copula functions. A two-stage Bayesian estimation method is introduced to facilitate the parameter estimation with incomplete or complete degradation observations. Degradation inferences for missing observations and future observations are developed separately. The capability of the proposed method for degradation inference is verified through a simulation study. An illustrative example characterized as bivariate s-dependent degradation processes with incomplete degradation observations is presented to illustrate the proposed method. The proposed model can model various types of performance indicators, in addition to the indicators with linear degradation paths. Moreover, the proposed method for degradation inference has the merit of leveraging the parameter estimations of the marginal degradation processes, and the analytical dependence relationship provided by conditional copula functions. Both of these advantages can facilitate the degradation inferences, reliability estimation, and RUL prediction, which are critical for the decision making for the preventive maintenance and system health management of complex systems.

It is worth mentioning that the proposed method can be extended to the situation of multivariate degradation processes by properly substituting the bivariate copula with a multivariate copula. In addition, a prerequisite for the degradation inference of a missing degradation observation is that at least one degradation process is observed at this particular observation point. Our future work will focus on model comparisons among various types of bivariate degradation process models. The selection of marginal degradation processes and copula functions is also of interest. Another possible research will focus on degradation test planning with the proposed bivariate degradation process models considering the inherent unit-to-unit variability, the external measurement error, and the measurement intervals.

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