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Uncertain dynamic responses of imprecisely defined arbitrary order fractionally damped beam subject to various loads

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Abstract

Purpose – Recently, fractional differential equations have been used to model various physical and engineering problems. One may need a reliable and efficient numerical technique for the solution of these types of differential equations, as sometimes it is not easy to get the analytical solution. However, in general, in the existing investigations, involved parameters and variables are defined exactly, whereas in actual practice it may contain uncertainty because of error in observations, maintenance induced error, etc. Therefore, the purpose of this paper is to find the dynamic response of fractionally damped beam approximately under fuzzy and interval uncertainty.

Design/methodology/approach – Here, a semi analytical approach, variational iteration method (VIM), has been considered for the solution. A newly developed form of fuzzy numbers known as double parametric form has been applied to model the uncertainty involved in the system parameters and variables.

Findings – VIM has been successfully implemented along with double parametric form of fuzzy number to find the uncertain dynamic responses of the fractionally damped beam. The advantage of this approach is that the solution can be written in power series or compact form. Also, this method converges rapidly to have the accurate solution. The uncertain responses subject to impulse and step loads have also been computed and the behaviours of the responses are analysed. Applying the double parametric form, it reduces the computational cost without separating the fuzzy equation into coupled differential equations as done in traditional approaches.

Originality/value – Uncertain dynamic responses of fuzzy fractionally damped beam using the newly developed double parametric form of fuzzy numbers subject to unit step and impulse loads have been obtained. Gaussian fuzzy numbers are used to model the uncertainties. In the methodology using the alpha cut form, corresponding beam equation is first converted to an interval-based fuzzy equation. Next, it has been transformed to crisp form by applying double parametric form of fuzzy numbers. Finally, VIM has been applied to solve the same for the general fuzzy responses. Various numerical examples have been taken in to consideration.

Keywords Double parametric form, Fractionally damped beam, Gaussian fuzzy number, Unit impulse function, Unit step function

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1. Introduction

In the recent years, arbitrary order (fractional) differential equations have been used to model various physical and engineering problems. Many important works have been reported regarding fractional calculus in the past few decades. Related to this field, several excellent books (Samko *et al.*, 1993; Miller and Ross, 1993; Oldham and Spanier, 1974; Kiryakova, 1993 and Podlubny, 1999; Kilbas *et al.*, 2006) have also been written by different authors. These books give an extensive review on this field which may help the reader to understand the basic concepts of fractional calculus. The advantages of fractional order derivatives considering the differential equations has been excellently described by Almeida *et al.* (2016). They have mentioned that fractional differential equations can model various problems with high-order dynamics and complex nonlinear phenomena more efficiently than general differential equations. It occurs mainly for two reasons; first, the freedom is to choose the arbitrary order of the derivative and integral operators without restricting the integer-order only. Second, fractional order derivatives depend on not only local conditions but also the past. By considering fractional derivatives, one may have an infinite choice of derivative orders for considerations, and then it is easy to find the fractional differential equation which describes the dynamics of the model in a better way. In this regard, various examples and experimental data have been considered by Almeida *et al.* (2016). Also, they have observed that experimental data, where non-integer derivatives allow the solution curve to model more efficiently the problems. Moreover, Bagley and Torvik (1983a, 1983b, 1985) have shown that the viscoelastic damping of structures defined by fractional derivatives has various important features. They explained that these models are based on accepted molecular theories which govern the mechanical behaviour of the media. Also, they have given that the second law of thermodynamics are satisfied by these models. Finally, only few parameters are required to define the viscoelastic properties of these models.

However, many authors have developed various methods to solve arbitrary order ordinary and partial differential equations and integral equations of physical systems. Suarez and Shokooh (1997) used an eigenvector expansion method for the solution of a mechanical spring-mass system containing fractional derivatives, and the results obtained are found quite satisfactory. The same type of problem is also studied by Yuan and Agrawal (2002) using a numerical technique when the damping factor is defined as fractional. Recently, Behera and Chakraverty (2013c, 2015) studied numerical solution of fractionally damped beam using homotopy perturbation method. Also, dynamic responses of fractionally damped spring mass system have been investigated by Chakraverty and Behera (2013). Very recently, Escalante-Martínez *et al.* (2016) have presented theoretical as well as experimental approach to find the viscous damping coefficient in the spring-mass viscodamper system. There, the authors have analysed the nonlocal damping model considering fractional derivatives. Different theories and applications of fractional differential equations can be found in Agrawal (2004), Odibat and Momani (2008), Jumarie (2009), Wei *et al.* (2010), Qian *et al.* (2010), Rahimy (2010), Gómez-Aguilar *et al.* (2016, 2017) and Yépez-Martínez *et al.* (2016).

In general, system parameters, initial and boundary conditions involved in the modelling problems are considered as deterministic or defined exactly. But, rather than the deterministic value, we may have only the vague, imprecise and incomplete information about the variables and parameters being a result of errors in measurement, observations, experiment, applying different operating conditions or it may be maintenance induced error, etc. which are uncertain in nature. Basically, these uncertainties can be modelled through probabilistic, interval and fuzzy theory. In probabilistic practice, the variables of uncertain nature are assumed as random variables with joint probability density functions.

Unfortunately, probabilistic methods may not able to deliver reliable results at the required precision without sufficient experimental data. It may be because of the probability density functions involved in it. As such, in the recent decades, interval analysis and fuzzy theory are becoming powerful tools for many real-life applications for uncertainty modelling. In these approaches, the uncertain variables and parameters are represented by interval and fuzzy numbers, vectors or matrices. Moreover, intervals are subcase of fuzzy numbers. Hence, we have used fuzzy set theory and interval solutions are obtained as special cases.

Fuzzy set theoretical concept was first developed by Zadeh (1965), which has been further used for the uncertainty analysis of various problems (Hanss and Turrin, 2010; Rao *et al.*, 2010; Farkas *et al.*, 2012; Behera and Chakraverty, 2013a, 2013b; Tapaswini and Chakraverty, 2014a, 2014b; Tapaswini *et al.*, 2015) in a wide range. As both fractional and fuzzy play a vital role in the modelling and design process, various attempts have been made to combine the both. In this regard, some recent contributions on the theory of fuzzy fractional modelling may be seen as follows. The concept of fuzzy fractional differential equation was introduced recently by Agrawal *et al.* (2010). Fractional differential equation with the fuzzy initial condition has been investigated by Arshad and Lupulescu (2011). Differential transform method is applied by Mohammed *et al.* (2011) for solution of fuzzy fractional initial value problems. Boundary value problem for fuzzy fractional differential equations with finite delay has been solved by Wang and Liu (2011). Salahshour *et al.* (2012) developed Riemann–Liouville differentiability by using Hukuhara difference called Riemann–Liouville H-differentiability and solved fuzzy fractional differential equations by Laplace transform. Existence and uniqueness of the solution is studied by Karthikeyan and Chandran (2011) for functional fractional fuzzy impulsive differential equations. Jeong (2010) discussed existence and uniqueness results for fuzzy fractional differential equations with infinite delay. However, Ahmad *et al.* (2013) implemented Zadeh's extension principle for solving fuzzy fractional differential equations. Very recently, Chakraverty and Behera (2015) investigated the solution procedure for the uncertain dynamic responses of fuzzy fractionally damped discrete system called a mechanical spring-mass system.

Recently, variational iteration method (VIM) is found to be a powerful tool for the analysis of linear and non-linear physical problems. VIM was first developed by He (1999, 2000) and was successfully applied to solve various linear and non-linear differential equations of scientific and engineering problems. Very recently, VIM has been applied to a wide class of other physical problems (Maidi and Corriou, 2013; Wazwaz, 2009; Momani *et al.*, 2006; Huang and Liu, 2013; Hemeda, 2008; He, 2007; Abulwafa *et al.*, 2008).

In the present analysis, VIM is applied for the numerical solution of uncertain dynamic response of a fuzzy fractionally damped beam with fuzzy initial condition. Uncertainty in the initial condition is defined in term of fuzzy numbers. In the solution procedure, newly developed double parametric form of fuzzy numbers are used. Unit step and impulse loads are considered for the present analysis. Literature review reveals that fuzzy differential equations are always converted to two crisp differential equations in general to obtain the solution bounds. But in the proposed methodology, the fuzzy differential equation has been converted to a single crisp differential equation using double parametric form of fuzzy numbers (Behera *et al.*, 2015) and then the corresponding crisp differential equation is solved to obtain the final fuzzy solution by substituting the parametric values.

This paper is organized as follows. In Section 2, some basic preliminaries related to the present investigation are given. Basic idea of VIM has been discussed in Section 3. Followed by this in Section 4, VIM has been applied with the proposed technique to solve the fuzzy fractionally damped beam. After that the uncertain responses subjected to unit step and impulse loads have been analysed in Section 5. Next, numerical results and discussion for various parameters are presented in Section 6. In the last section conclusion has been drawn.

2. Preliminaries

In this section, we present some notations, definitions (Ross, 2004; Zimmermann, 2001) which are used further in this paper.

Definition 2.1 Fuzzy set

A fuzzy set $\tilde{U}$ on the real line R is defined as the set of ordered pairs such that:

$$\tilde{U} = \{(x, \mu_{\tilde{U}}(x)) | x \in R, \mu_{\tilde{U}}(x) \in [0, 1]\}$$

where, $\mu_{\tilde{U}}(x)$ is called the membership function.

Definition 2.2 Fuzzy number

A fuzzy number $\tilde{U}$ is convex normalised fuzzy set $\tilde{U}$ of the real line R such that:

$$\{\mu_{\tilde{U}}(x) : R \to [0, 1], \forall x \in R\}$$

where, $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set and it is piecewise continuous.

Definition 2.3 Gaussian Fuzzy Number (GFN)

Let us now define an arbitrary asymmetrical Gaussian fuzzy number, $\tilde{U} = (\delta, \sigma_l, \sigma_r)$. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ will be as follows:

$$\mu_U(x) = \begin{cases} \exp\left[-(x-\delta)^2/2\sigma_l^2\right] & \text{for } x \le \delta \\ \exp\left[-(x-\delta)^2/2\sigma_r^2\right] & \text{for } x \ge \delta \end{cases} \quad \forall\, x \in R$$

where, the modal value is denoted as δ and σ_l, σ_r denote the left-hand and right-hand spreads (fuzziness) corresponding to the Gaussian distribution. For symmetric Gaussian fuzzy number, the left-hand and right-hand spreads are equal, i.e. $\sigma_l = \sigma_r = \sigma$. So the symmetric Gaussian fuzzy number may be written as $\tilde{U} = (\delta, \sigma, \sigma)$ and corresponding membership function may be defined as $\mu_{\tilde{U}}(x) = \exp\{-\beta(x-\delta)^2\}$ $\forall\, x \in R$ where, $\eta = 1/2\sigma^2$. The symmetric Gaussian fuzzy number in parametric can be represented as:

$$\tilde{U} = [\underline{u}(r), \overline{u}(r)] = \left[\delta - \sqrt{-\frac{(\log_e r)}{\eta}}, \delta + \sqrt{-\frac{(\log_e r)}{\eta}}\right], \text{ where, } r \in [0, 1]$$

Definition 2.4 Single parametric form of fuzzy numbers

The triangular fuzzy number $\tilde{U} = (a, b, c)$ can be represented with an ordered pair of functions through r – cut approach as:

$$\tilde{U}(r) = [\underline{u}(r), \overline{u}(r)] = [(b-a)r + a, \ -(c-b)r + c] \text{ where, } r \in [0, 1]$$

The r – cut form is known as parametric form or single parametric form of fuzzy numbers.

It may be noted that the lower and upper bounds of the fuzzy numbers satisfy the following requirements:

- $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1].
- $\bar{u}(r)$ is a bounded right continuous non-increasing function over [0, 1].
- $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

Definition 2.5 Double parametric form of fuzzy number

Using the parametric form as discussed in Definition 2.4, we have $\tilde{U}(r) = [\underline{u}(r), \overline{u}(r)]$. Now one may write this as crisp number with double parametric form:

$$\tilde{U}(r,\ \beta) = \beta(\overline{u}(r) - \underline{u}(r)) + \underline{u}(r) \text{ where } \beta \in [0,\ 1].$$

It is worth mentioning that the lower and upper bounds in single parametric form may be obtained if we put $\beta = 0$ and 1 respectively in the above double parametric form. This may be represented as $\tilde{U}(r,\ 0) = \underline{u}(r)$ and $\tilde{U}(r, 1) = \bar{u}(r)$.

Definition 2.6 Fuzzy arithmetic

For any two arbitrary fuzzy numbers $\tilde{x} = [\underline{x}(r), \overline{x}(r)], \tilde{y} = [\underline{y}(r), \overline{y}(r)]$ and scalar k, the fuzzy arithmetic is defined as follows:

- $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$:
- $\tilde{x} + \tilde{y} = [\underline{x}(r) + \underline{y}(r),\ \overline{x}(r) + \overline{y}(r)]$
- $\tilde{x} \times \tilde{y} = [\min(S), \max(S)]$ where: $S = \{\underline{x}(r) \times \underline{y}(r), \underline{x}(r) \times \overline{y}(r), \overline{x}(r) \times \underline{y}(r), \overline{x}(r) \times \overline{y}(r)\}$
- $k\tilde{x} = \begin{cases} [k\overline{x}(r), k\underline{x}(r)], k < 0 \\ [k\underline{x}(r), k\overline{x}(r)], k \geq 0 \end{cases}$

3. Basic idea of variational iteration method (He, 1999, 2000)

To illustrate the basic idea of the technique, we consider the following general nonlinear system:

$$L[s(t)] + N[s(t)] = z(t) \tag{1}$$

where, L is a linear operator, N is a nonlinear operator and $z(t)$ is a given continuous function.

The basic character of the method is to construct a correction functional for equation (1) as follows:

$$s_{n+1}(t) = s_n(t) + \int_0^t \lambda(\tau)\{Ls_n(\tau) + N\hat{s}_n(\tau) - z(\tau)\}d\tau \tag{2}$$

where λ is a general Lagrangian multiplier which can be identified via variational theory. s_n is the nth approximate solution and $\hat{s}_n$ denotes a restricted variation, i.e. $\delta\hat{s}_n = 0$. The initial approximation s_0 can be freely chosen if it satisfies the initial and boundary conditions of the problem. However, the success of the method depends on the proper selection of the initial approximation s_0. We approximate the solution as:

$$s(x, t) = \lim_{n \to \infty} s_n(x, t) \tag{3}$$

4. Solution to uncertain fractionally damped viscoelastic beam

To develop numerical schemes for a fuzzy fractionally damped viscoelastic beam let us consider a fuzzy linear differential equation which describes the dynamics of the system with the damping as an arbitrary fractional derivative of order α:

$$\rho A \frac{\partial^2 \tilde{v}}{\partial t^2} + c \frac{\partial^\alpha \tilde{v}}{\partial t^\alpha} + EI \frac{\partial^4 \tilde{v}}{\partial x^4} = F(x,t) \tag{4}$$

where ρ, A, c, E and I represents the mass density, cross-sectional area, damping coefficients per unit length, Young's modulus of elasticity and moment of inertia of the beam, respectively. $F(x, t)$ is the externally applied force and $\tilde{v}(x,t)$ is the transverse fuzzy displacement. $\frac{\partial^\alpha}{\partial t^\alpha}$ is the fractional derivative of order $\alpha \in [0, 1]$ of the fuzzy displacement function $\tilde{v}(x,t)$. Initial conditions are considered as fuzzy viz. $\tilde{v}(0) = \tilde{v}'(0) = (0,\ 0.1,\ 0.1)$.

In the solution procedure, first, the above equation is converted to an interval based fuzzy fractional differential equation using single parametric form. Then by using double parametric form, interval based fuzzy fractional differential equation is reduced to a crisp differential equation. Next, VIM is applied to solve the corresponding differential equation.

Equation (4) may be written as:

$$\frac{\partial^2 \tilde{v}}{\partial t^2} + \frac{c}{\rho A}\frac{\partial^\alpha \tilde{v}}{\partial t^\alpha} + \frac{EI}{\rho A}\frac{\partial^4 \tilde{v}}{\partial x^4} = \frac{F(x,t)}{\rho A} \tag{5}$$

As per the single parametric form, we may write equation (5) as:

$$\left[\frac{\partial^2 \underline{v}(x,t;r)}{\partial t^2}, \frac{\partial^2 \overline{v}(x,t;r)}{\partial t^2}\right] + \frac{c}{\rho A}\left[\frac{\partial^\alpha \underline{v}(x,t;r)}{\partial t^\alpha}, \frac{\partial^\alpha \overline{v}(x,t;r)}{\partial t^\alpha}\right]$$
$$+ \frac{EI}{\rho A}\left[\frac{\partial^4 \underline{v}(x,t;r)}{\partial x^4}, \frac{\partial^4 \overline{v}(x,t;r)}{\partial x^4}\right] = \frac{F(x,t)}{\rho A} \tag{6}$$

subject to fuzzy initial condition:

$$[\underline{v}(x,0;r), \overline{v}(x,0;r)] = [\underline{v}'(x,0;r), \overline{v}'(x,0;r)]\left[-\sqrt{-0.02\log r}, \sqrt{-0.02\log r}\right]$$

where, $r \in [0,1]$.

Next, using the double parametric form (as discussed in Definition 2.5) equation (6) can be expressed as:

$$\left\{\beta\left(\frac{\partial^2 \overline{v}(x,t;r)}{\partial t^2} - \frac{\partial^2 \underline{v}(x,t;r)}{\partial t^2}\right) + \frac{\partial^2 \underline{v}(x,t;r)}{\partial t^2}\right\}$$
$$+ \frac{c}{\rho A}\left\{\beta\left(\frac{\partial^\alpha \overline{v}(x,t;r)}{\partial t^\alpha} - \frac{\partial^\alpha \underline{v}(x,t;r)}{\partial t^\alpha}\right) + \frac{\partial^\alpha \underline{v}(x,t;r)}{\partial t^\alpha}\right\}$$
$$+ \frac{EI}{\rho A}\left\{\beta\left(\frac{\partial^4 \overline{v}(x,t;r)}{\partial x^4} - \frac{\partial^4 \underline{v}(x,t;r)}{\partial x^4}\right) + \frac{\partial^4 \underline{v}(x,t;r)}{\partial x^4}\right\} = \frac{F(x,t)}{\rho A} \tag{7}$$

subject to fuzzy initial conditions.

$$\{\beta(\overline{v}(x,0;r)-\underline{v}(x,0;r))+\underline{v}(x,0;r)\}=\left\{\beta\left(2\sqrt{-0.02\log r}\right)-\sqrt{-0.02\log r}\right\}\text{ and}$$

$$\{\beta(\overline{v}'(x,0;r)-\underline{v}'(x,0;r))+\underline{v}'(x,0;r)\}=\left\{\beta\left(2\sqrt{-0.02\log r}\right)-\sqrt{-0.02\log r}\right\}$$

where, $\beta\in[0,1]$

Let us now denote:

$$\left\{\beta\left(\frac{\partial^2\overline{v}(x,t;r)}{\partial t^2}-\frac{\partial^2\underline{v}(x,t;r)}{\partial t^2}\right)+\frac{\partial^2\underline{v}(x,t;r)}{\partial t^2}\right\}=\frac{\partial^2\tilde{v}(x,t;r,\beta)}{\partial t^2},$$

$$\left\{\beta\left(\frac{\partial^\alpha\overline{v}(x,t;r)}{\partial t^\alpha}-\frac{\partial^\alpha\underline{v}(x,t;r)}{\partial t^\alpha}\right)+\frac{\partial^\alpha\underline{v}(x,t;r)}{\partial t^\alpha}\right\}=\frac{\partial^\alpha\tilde{v}(x,t;r,\beta)}{\partial t^\alpha},$$

$$\left\{\beta\left(\frac{\partial^4\overline{v}(x,t;r)}{\partial x^4}-\frac{\partial^4\underline{v}(x,t;r)}{\partial x^4}\right)+\frac{\partial^4\underline{v}(x,t;r)}{\partial x^4}\right\}=\frac{\partial^4\tilde{v}(x,t;r,\beta)}{\partial x^4},$$

$$\{\beta(\overline{v}(x,0;r)-\underline{v}(0;r))+\underline{v}(x,0;r)\}=\tilde{v}(x,0;r,\beta)\text{ and}$$

$$\{\beta(\overline{v}'(x,0;r)-\underline{v}'(x,0;r))+\underline{v}'(x,0;r)\}=\tilde{v}'(x,0;r,\beta)$$

Substituting these values in equation (7) we get:

$$\frac{\partial^2\tilde{v}(x,t;r,\beta)}{\partial t^2}+\frac{c}{\rho A}\frac{\partial^\alpha\tilde{v}(x,t;r,\beta)}{\partial t^\alpha}+\frac{EI}{\rho A}\frac{\partial^4\tilde{v}(x,t;r,\beta)}{\partial x^4}=\frac{F(x,t)}{\rho A} \tag{8}$$

with initial conditions $\tilde{v}(x,0;r,\beta)=\tilde{v}'(x,0;r,\beta)=\left\{\beta\left(2\sqrt{-0.02\log r}\right)-\sqrt{-0.02\log r}\right\}$.

By solving the corresponding equation (8) one may get the uncertain transverse fuzzy displacement$\tilde{v}(x,t;r,\beta)$. To obtain the lower and upper bounds of the solution in single parametric form one may substitute $\beta=0$ and 1 respectively and thus the solution bounds may be represented as:

$$\tilde{v}(x,t;r,0)=\underline{v}(x,t,r)\text{ and }\tilde{v}(x,t;r,1)=\overline{v}(x,t,r) \tag{9}$$

4.1. Application of VIM in to double parametric based form

First, we have applied VIM to solve equation (8). According to VIM, we may construct a correction functional as follows:

$$\tilde{v}_{n+1}(x,t;r,\beta)=\tilde{v}_n(x,t;r,\beta)$$
$$+\int_0^t\lambda(\tau)\left\{\frac{\partial^2}{\partial\tau^2}\tilde{v}_n(x,\tau;r,\beta)+\frac{c}{\rho A}\frac{\partial^\lambda}{\partial\tau^\lambda}\hat{\tilde{v}}_n(x,\tau;r,\beta)+\frac{EI}{\rho A}\frac{\partial^4}{\partial x^4}\hat{\tilde{v}}_n(x,\tau;r,\beta)-\frac{F(x,t)}{\rho A}\right\}d\tau \tag{10}$$

Making the above correction functional, i.e. equation (10) stationary, and noticing that $\delta\hat{\tilde{v}}_n = 0$, we obtain:

$$\delta\tilde{v}_{n+1}(x,t;r,\beta) = \delta\tilde{v}_n(x,t;r,\beta)$$
$$+\delta\int_0^t \lambda(\tau)\left\{\frac{\partial^2}{\partial\tau^2}\tilde{v}_n(x,\tau;r,\beta)+\frac{c}{\rho A}\frac{\partial^\alpha}{\partial\tau^\alpha}\hat{\tilde{v}}_n(x,\tau;r,\beta)+\frac{EI}{\rho A}\frac{\partial^4}{\partial x^4}\tilde{v}_n(x,\tau;r,\beta)-\frac{F(x,t)}{\rho A}\right\}d\tau$$
$$=\left(1-\lambda'(\tau)\right)\delta\tilde{v}_n(x,t;r,\beta)+\lambda(\tau)\delta\tilde{v}_n'(x,t;r,\beta)+\int_0^t \lambda''(\tau)\delta\tilde{v}_n(x,\tau;r,\beta)d\tau = 0 \tag{11}$$

Thus, we obtain the Euler–Lagrange equation:

$$\lambda''(\tau) = 0 \tag{12}$$

with natural boundary:

$$\begin{gathered} 1-\lambda'(\tau) = 0, \\ \lambda(\tau) = 0 \end{gathered} \tag{13}$$

So, the Lagrange multiplier can be easily identified as follows:

$$\lambda = \tau - t \tag{14}$$

Substituting the identified Lagrange multiplier into equation (10), following variational iteration formula can be obtained as:

$$\tilde{v}_{n+1}(x,t;r,\beta) = \tilde{v}_n(x,t;r,\beta)$$
$$+\int_0^t (\tau - t)\left\{\frac{\partial^2}{d\tau^2}\tilde{v}_n(x,\tau;r,\beta)+\frac{c}{\rho A}\frac{\partial^\alpha}{d\tau^\alpha}\hat{\tilde{v}}_n(x,\tau;r,\beta)+\frac{EI}{\rho A}\frac{\partial^4}{dx^4}\tilde{v}_n(x,\tau;r,\beta)-\frac{F(x,t)}{\rho A}\right\}d\tau \tag{15}$$

5. Uncertain response analysis

Let us consider the external applied force $F(x, t)$ as:

$$F(x,t) = f(x)g(t) \tag{16}$$

where $f(x)$ is a specified space dependent deterministic function and $g(t)$ is time dependent process. In the following paragraph, we will examine the fuzzy response of the dynamic system (8) subject to unit step and impulse loading conditions.

5.1. Unit step function response

We will now consider the response of the fuzzy fractionally damped beam subject to a unit step load of the form $g(t) = Bu(t)$ where $u(t)$ is the Heaviside function and B is a constant.

We start with an initial approximation:

$$\tilde{v}_0 = \tilde{v}(x, 0, r, \beta) = (1+t)\left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\} \tag{17}$$

and from variational iterational equation (15) with the application of Caputo derivative as defined in Chakraverty and Behera (2013), Behera and Chakraverty (2013c) for fractional differentiation, we have:

$$\tilde{v}_1(x, t; r, \beta) = \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}\left(1 + t - \frac{c}{\rho A}\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right) + \frac{fBt^2}{2\rho A} \tag{18}$$

$$\begin{aligned}\tilde{v}_2(x, t; r, \beta) &= \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}\\ &\quad\times\left(1 + t - \frac{c}{\rho A}\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{c^2}{\rho^2A^2}\frac{t^{5-2\alpha}}{\Gamma(6-2\alpha)}\right)\\ &\quad - \frac{fBc}{\rho^2A^2}\frac{t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{EIBf^4}{\rho^2A^2}\frac{t^4}{\Gamma(5)} + \frac{fBt^2}{2\rho A}\end{aligned} \tag{19}$$

$$\begin{aligned}\tilde{v}_3(x, t; r, \beta) &= \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}\\ &\quad\times\left(1 + t - \frac{c}{\rho A}\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{c^2}{\rho^2A^2}\frac{t^{5-2\alpha}}{\Gamma(6-2\alpha)} - \frac{c^3}{\rho^3A^3}\frac{t^{7-3\alpha}}{\Gamma(8-3\alpha)}\right)\\ &\quad + \frac{fBc^2}{\rho^3A^3}\frac{t^{6-2\alpha}}{\Gamma(7-2\alpha)} - \frac{fBc}{\rho^2A^2}\frac{t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{EIBf^4}{\rho^2A^2}\frac{t^4}{\Gamma(5)}\\ &\quad + \frac{2EIBcf^4}{\rho^3A^3}\frac{t^{6-\alpha}}{\Gamma(7-\alpha)} + \frac{E^2I^2Bf^8}{\rho^3A^3}\frac{t^6}{\Gamma(7)} + \frac{fBt^2}{2\rho A}\end{aligned} \tag{20}$$

$$\begin{aligned}\tilde{v}_4(x, t; r, \beta) &= \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}\\ &\quad\times\left(\begin{aligned}&1 + t - \frac{c}{\rho A}\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{c^2}{\rho^2A^2}\frac{t^{5-2\alpha}}{\Gamma(6-2\alpha)}\\ &- \frac{c^3}{\rho^3A^3}\frac{t^{7-3\alpha}}{\Gamma(8-3\alpha)} - \frac{c^4}{\rho^4A^4}\frac{t^{9-4\alpha}}{\Gamma(10-4\alpha)}\end{aligned}\right) - \frac{fBc^3}{\rho^4A^4}\frac{t^{8-3\alpha}}{\Gamma(9-3\alpha)}\end{aligned}$$

$$+\frac{fBc^2}{\rho^3A^3}\frac{t^{6-2\alpha}}{\Gamma(7-2\alpha)}-\frac{fBc}{\rho^2A^2}\frac{t^{4-\alpha}}{\Gamma(5-\alpha)}-\frac{EIBf^4}{\rho^2A^2}\frac{t^4}{\Gamma(5)}$$
$$+\frac{2EIBcf^4}{\rho^3A^3}\frac{t^{6-\alpha}}{\Gamma(7-\alpha)}-\frac{3EIBc^2f^4}{\rho^4A^4}\frac{t^{8-2\alpha}}{\Gamma(9-2\alpha)}+\frac{E^2I^2Bf^8}{\rho^3A^3}\frac{t^6}{\Gamma(7)}$$
$$-\frac{3E^3I^3Bf^{12}}{\rho^4A^4}\frac{t^8}{\Gamma(9)}-\frac{3E^2I^2Bcf^8}{\rho^4A^4}\frac{t^{8-\alpha}}{\Gamma(9-\alpha)}+\frac{fBt^2}{2\rho A} \quad (21)$$

and so on, where $f^{(i)}=\frac{\partial^i f}{\partial x^i}$.

Explaining in details to get the above expressions, first for equation (18), we have to substitute the value of the initial approximation $\tilde{v}_0(x,t;r,\beta)$ as defined in equation (17) into equation (15).

Accordingly, one may have:

$$\tilde{v}_1(x,t;r,\beta)=(1+t)\left\{\beta\left(2\sqrt{-0.02\log r}\right)-\sqrt{-0.02\log r}\right\}+\int_0^t(\tau-t)\{\varphi\}d\tau$$

where:

$$\varphi=\frac{\partial^2}{d\tau^2}\left[(1+t)\left\{\beta\left(2\sqrt{-0.02\log r}\right)-\sqrt{-0.02\log r}\right\}\right]$$
$$+\frac{c}{\rho A}\frac{\partial^\alpha}{d\tau^\alpha}\left[(1+t)\left\{\beta\left(2\sqrt{-0.02\log r}\right)-\sqrt{-0.02\log r}\right\}\right]$$
$$+\frac{EI}{\rho A}\frac{\partial^4}{dx^4}\left[(1+t)\left\{\beta\left(2\sqrt{-0.02\log r}\right)-\sqrt{-0.02\log r}\right\}\right]-\frac{fBu(t)}{\rho A}$$

Simplifying the above expression for $\tilde{v}_1(x,t;r,\beta)$ using the definition for fractional derivative in Caputo sense as defined in Chakraverty and Behera (2013), Behera and Chakraverty (2013c) for the second term and by considering general differentiation for first and third term involved in expression of φ we have:

$$\tilde{v}_1(x,t;r,\beta)=(1+t)\left\{\beta\left(2\sqrt{-0.02\log r}\right)-\sqrt{-0.02\log r}\right\}$$
$$+\int_0^t(\tau-t)\left(\frac{c}{\rho A}\frac{\left\{\beta\left(2\sqrt{-0.02\log r}\right)-\sqrt{-0.02\log r}\right\}\Gamma(2)\tau^{1-\alpha}}{\Gamma(2-\alpha)}-\frac{fBu(t)}{\rho A}\right)d\tau$$

By general differentiation, the first and third term involved in expression of φ vanishes.

(For better understanding for the Caputo's derivative from definition (Podlubny, 1999), we have:

$$D^\alpha C = 0, \qquad (C \text{ is a constant})$$

$$D^\alpha t^\gamma = \begin{cases} 0, & \gamma \le \alpha - 1 \\ \dfrac{\Gamma(\gamma+1)t^{\gamma-\alpha}}{\Gamma(\gamma-\alpha+1)}, & \gamma > \alpha - 1 \end{cases}$$

here D^α is the fractional derivative of order α.)

Next, again the above expression for $\tilde{v}_1(x,t;r,\beta)$ has been simplified as below substituting $\Gamma(2) = 1$ and accordingly one may have:

$$\begin{aligned} \tilde{v}_1(x,t;r,\beta) &= (1+t)\left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\} \\ &\quad + \int_0^t \tau\left(\frac{c}{\rho A}\frac{\left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}\tau^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{fBu(t)}{\rho A}\right)d\tau \\ &\quad - \int_0^t t\left(\frac{c}{\rho A}\frac{\left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}\tau^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{fBu(t)}{\rho A}\right)d\tau \end{aligned}$$

After applying the simple integration rule for the integrating term involved in the above equation, it gives:

$$\begin{aligned} \tilde{v}_1(x,t;r,\beta) &= (1+t)\left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\} \\ &\quad + \frac{c}{\rho A}\frac{\left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}t^{3-\alpha}}{(3-\alpha)\Gamma(2-\alpha)} - \frac{fBt^2}{2\rho A} \\ &\quad - \frac{c}{\rho A}\frac{\left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}t^{3-\alpha}}{(2-\alpha)\Gamma(2-\alpha)} + \frac{fBt^2}{\rho A} \end{aligned}$$

Hence, equivalently the above expression can be represented as:

$$\tilde{v}_1(x,t;r,\beta) = \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}\left(1 + t - \frac{c}{\rho A}\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right) + \frac{fBt^2}{2\rho A}$$

This represents equation (18). Similarly, equations (19) to (21) and rest of the components can be obtained. Therefore, the solution can be written in general form as:

$$\begin{aligned} \tilde{v}(x,t;r,\beta) &= \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}\left\{1 + \sum_{k=0}^{\infty}\frac{t^{(2-\alpha)k+1}}{\Gamma((2-\alpha)k+2)}\right\} \\ &\quad + \frac{B}{\rho A}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left(\frac{EI}{\rho A}\right)^p f^{(4p)}t^{2(p+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+p)!t^{(2-\alpha)j}}{j!\Gamma\left((2-\alpha)j+2p+3\right)}. \end{aligned} \tag{22}$$

As discussed above, to obtain the solution bound in single parametric form we may put $\beta = 0$ and 1 to get the lower and upper bound of the solution, respectively. This may be represented as $\tilde{v}(x,t;r,0) = \underline{v}(x,t,r)$ and $\tilde{v}(x,t;r,1) = \overline{v}(x,t,r)$.

Hence,

$$\underline{v}(x,t;r,0) = \left(-\sqrt{-0.02\log r}\right)\left\{1 + \sum_{k=0}^{\infty}\frac{t^{(2-\alpha)k+1}}{\Gamma((2-\alpha)k+2)}\right\} + \frac{B}{\rho A}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left(\frac{EI}{\rho A}\right)^p f^{(4p)}t^{2(p+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+p)!\,t^{(2-\alpha)j}}{j!\,\Gamma\big((2-\alpha)j+2p+3\big)} \tag{23}$$

$$\overline{v}(x,t;r,1) = \left(\sqrt{-0.02\log r}\right)\left\{1 + \sum_{k=0}^{\infty}\frac{t^{(2-\alpha)k+1}}{\Gamma((2-\alpha)k+2)}\right\} + \frac{B}{\rho A}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left(\frac{EI}{\rho A}\right)^p f^{(4p)}t^{2(p+1)}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+p)!\,t^{(2-\alpha)j}}{j!\,\Gamma\big((2-\alpha)j+2p+3\big)}. \tag{24}$$

5.2. Unit impulse function response

Next, we consider response of the beam subject to a unit impulse load of the form $g(t) = \delta(t)$ where $\delta(t)$ is the unit impulse function. Using VIM in this case again, we have:

$$\tilde{v}_0(x,t;r,\beta) = \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}(1+t) \tag{25}$$

$$\tilde{v}_1(x,t;r,\beta) = \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}\left(1 + t - \frac{c}{\rho A}\frac{t^{3-\alpha}}{\Gamma(4-\alpha)}\right) + \frac{fBt}{\rho A} \tag{26}$$

$$\tilde{v}_2(x,t;r,\beta) = \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\} \times\left(1 + t - \frac{c}{\rho A}\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{c^2}{\rho^2A^2}\frac{t^{5-2\alpha}}{\Gamma(6-2\alpha)}\right) - \frac{fBc}{\rho^2A^2}\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{EIBf^4}{\rho^2A^2}\frac{t^3}{\Gamma(4)} + \frac{fBt}{\rho A} \tag{27}$$

$$\tilde{v}_3(x,t;r,\beta) = \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\} \times\left(1 + t - \frac{c}{\rho A}\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{c^2}{\rho^2A^2}\frac{t^{5-2\alpha}}{\Gamma(6-2\alpha)} - \frac{c^3}{\rho^3A^3}\frac{t^{7-3\alpha}}{\Gamma(8-3\alpha)}\right) + \frac{fBc^2}{\rho^3A^3}\frac{t^{5-2\alpha}}{\Gamma(6-2\alpha)} - \frac{fBc}{\rho^2A^2}\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{2EIBcf^4}{\rho^3A^3}\frac{t^{5-\alpha}}{\Gamma(6-\alpha)} + \frac{E^2I^2Bf^8}{\rho^3A^3}\frac{t^5}{\Gamma(6)} - \frac{EIBf^4}{\rho^2A^2}\frac{t^3}{\Gamma(4)} + \frac{fBt}{\rho A} \tag{28}$$

and so on, where $f^{(i)} = \frac{\partial^i f}{\partial x^i}$. In the similar manner, the rest of the components can be obtained. Therefore, the solution can be written in general form as:

$$\tilde{v}(x,t;r,\beta) = \left\{\beta\left(2\sqrt{-0.02\log r}\right) - \sqrt{-0.02\log r}\right\}\left\{1 + \sum_{k=0}^{\infty}\frac{t^{(2-\alpha)k+1}}{\Gamma((2-\alpha)k+2)}\right\} + \frac{1}{\rho A}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left(\frac{EI}{\rho A}\right)^p f^{(4p)}t^{2p+1}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+p)!\,t^{(2-\alpha)j}}{j!\,\Gamma\big((2-\alpha)j+2p+2\big)} \tag{29}$$

To obtain the solution bound in single parametric form we may put $\beta = 0$ and 1 to get the lower and upper bound of the solution, respectively. This may again be represented as $\tilde{v}(x,t;r,0) = \underline{v}(x,t,r)$ and $\tilde{v}(x,t;r,1) = \overline{v}(x,t,r)$. Hence:

$$\underline{v}(x,t;r,0) = \left(-\sqrt{-0.02\log r}\right)\left\{1 + \sum_{k=0}^{\infty}\frac{t^{(2-\alpha)k+1}}{\Gamma((2-\alpha)k+2)}\right\} + \frac{1}{\rho A}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left(\frac{EI}{\rho A}\right)^p f^{(4p)}t^{2p+1}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+p)!\,t^{(2-\alpha)j}}{j!\,\Gamma\big((2-\alpha)j+2p+2\big)} \tag{30}$$

$$\overline{v}(x,t;r,1) = \left(\sqrt{-0.02\log r}\right)\left\{1 + \sum_{k=0}^{\infty}\frac{t^{(2-\alpha)k+1}}{\Gamma((2-\alpha)k+2)}\right\} + \frac{1}{\rho A}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\left(\frac{EI}{\rho A}\right)^p f^{(4p)}t^{2p+1}\sum_{j=0}^{\infty}\left(\frac{-c}{\rho A}\right)^j\frac{(j+p)!\,t^{(2-\alpha)j}}{j!\,\Gamma\big((2-\alpha)j+2p+2\big)} \tag{31}$$

6. Numerical results and discussions

In this section, fuzzy responses subject to unit step and impulse function has been considered. Equations (23) and (30) provide the desired expressions for the considered loading condition. We have assumed a simply supported beam; hence, one may have $f(x) = \sin\left(\frac{\pi x}{L}\right)$. Here, the numerical computations have been done by truncating the infinite series in equations (23), (24), (30) and (31) to a finite number of terms. For numerical simulations, let us denote c/m and $EI/\rho A$, respectively, as $2\eta\omega_n^{3/2}$ and ω_n^2 where, ω_n is the natural frequency and η is the damping ratio. The values of the parameters are taken as $B = 1$, $\rho A = 1$, $L = \pi$, $x = 1/2$ and $m = 1$.

6.1 Case studies for fuzzy unit step response

Depending upon the natural frequency ω_n, damping ratio η and arbitrary order fractional derivative α subjected to unit step load, two different cases have been considered as follows. In the first case, the numerical values of the parameters are taken as $\omega_n = 5$rad/s, $\eta = 0.5$ and $\alpha = 0.2$. Next, in the second case, $\omega_n = 10$rad/s, $\eta = 0.05$ and $\alpha = 0.5$ have been considered. For first and second cases obtained fuzzy responses with respect to time are depicted in Figures 1 and 2, respectively. Next, Figures 3 and 4 give the effects of interval unit step responses for the particular value of membership r. Here, for $r = 1$, the lower and upper bound of the solution coincides each other and denoted as $\underline{v}(t;\,1) = \overline{v}(t;\,1) = v(t;\,1)$. Figure 3 represents the interval solution for $r = 0.4$ and 0.8 with $r = 1$ for the first case. Similarly, Figure 4 cites the results for the second case with $r = 1$.

Next, for the result analysis, we have considered the same parametric values as considered for Figure 3 by varying the fractional order derivative for damping factor. As regard, Figures 5 and 6 represent the interval unit step responses for $\alpha = 0.5$ and 0.8, respectively.

From the results, it can be seen that the uncertain width of the solution gradually decreases by increasing the membership value r. One may also observe from Figures 3, 5 and 6 that the oscillation of the uncertain bounds of the unit step responses gradually decreases by increasing the order of the fractional derivative.

To show the rapid convergence by this method. we have also incorporated various numerical simulation results by changing the truncation order (number of approximations) in Tables I to IV for unit step load. The data used in these tables are self-explanatory.

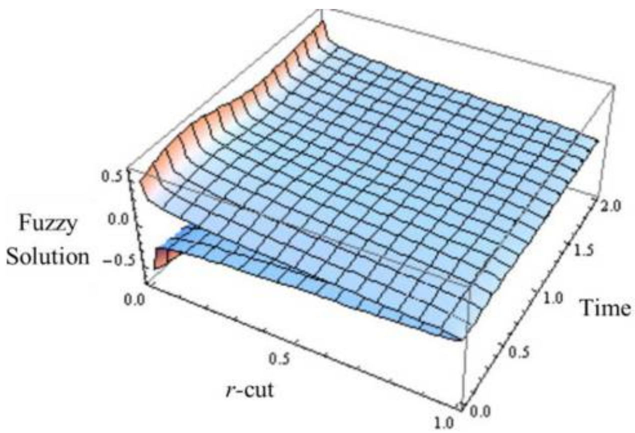

Figure 1. Fuzzy unit step response for $\omega_n = 5$ rad/s, $\eta = 0.5$ and $\alpha = 0.2$

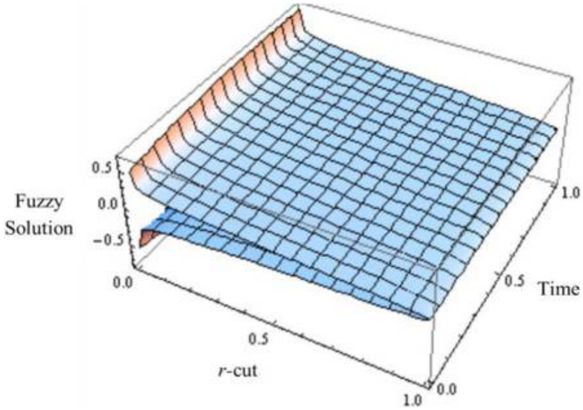

Figure 2.
Fuzzy unit step response for $\omega_n = 10$ rad/s, $\eta = 0.05$ and $\alpha = 0.5$

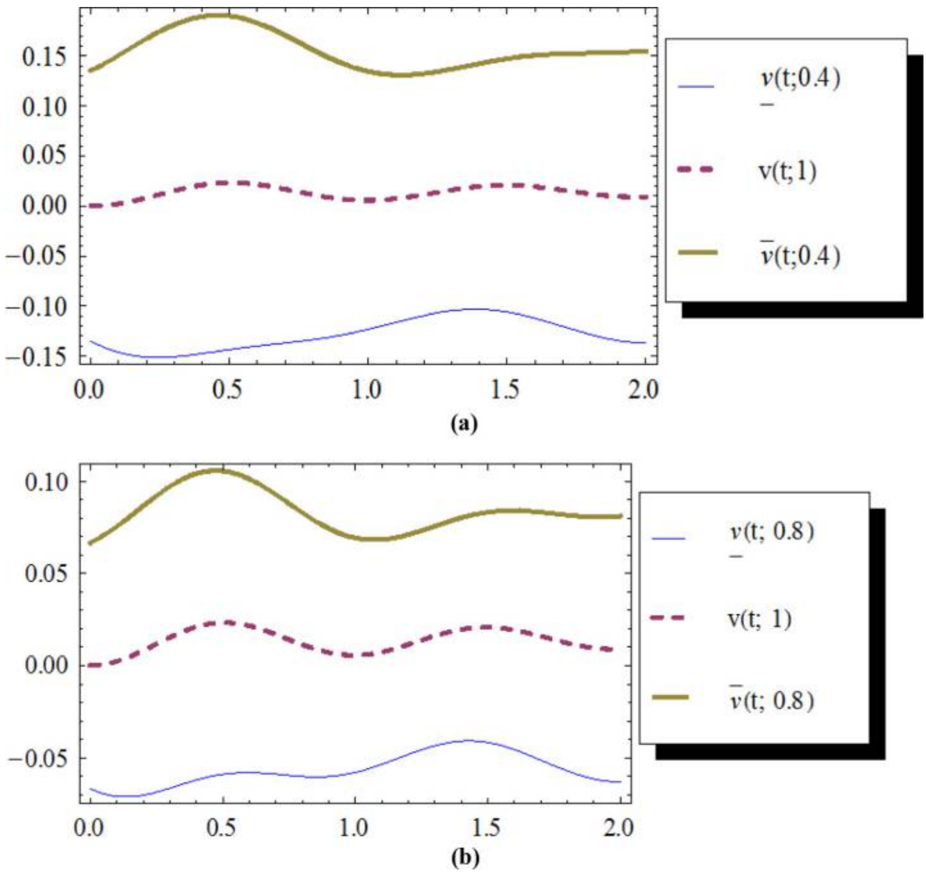

Figure 3.
Interval unit step response for (a) $r = 0.4$, (b) $r = 0.8$ with $\omega_n = 5$ rad/s, $\eta = 0.5$ $\alpha = 0.2$ and $r = 1$

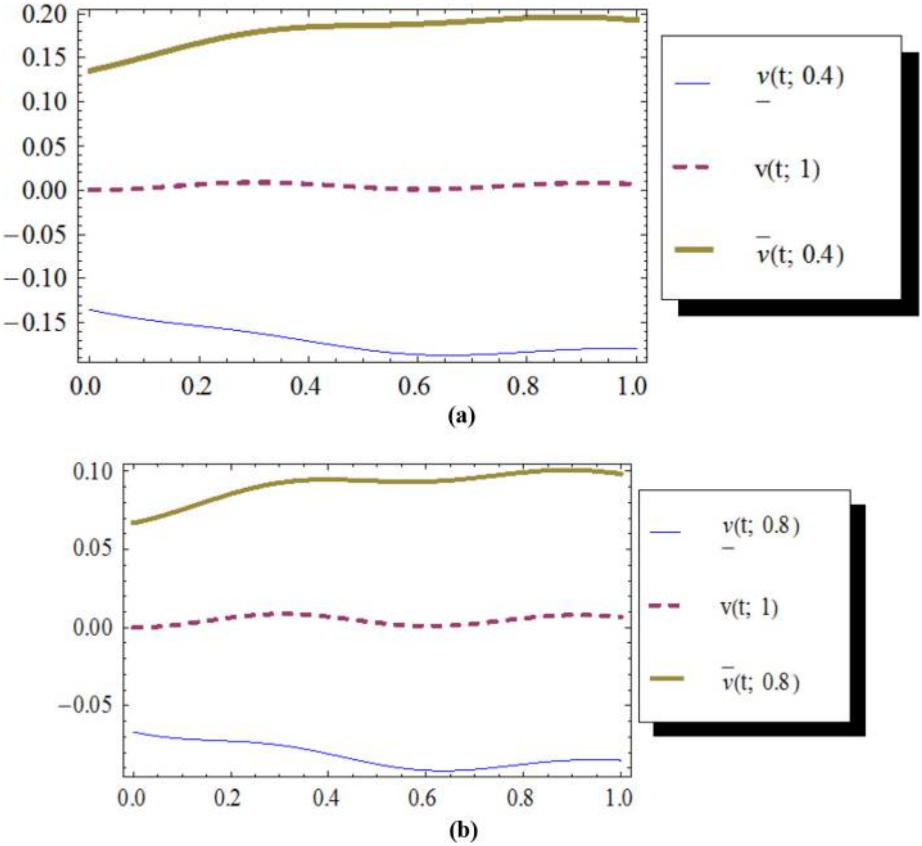

Figure 4. Interval unit step response for (a) $r = 0.4$, (b) $r = 0.8$ with $\omega_n = 10$ rad/s, $\eta = 0.05$, $\alpha = 0.5$ and $r = 1$

6.2 Case studies for fuzzy unit impulse response

Similarly, for the above case, here depending upon the system parameters, namely, natural frequency ω_n, damping ratio η and arbitrary order fractional derivative α, four different cases have been considered as follows subjected to unit impulse load. In the first case, the numerical values of the parameters are taken as ω_n = 5rad/s, η = 0.5 and α = 0.2. For the second case ω_n = 10rad/s, η = 0.5 and α = 0.5 have been considered. Next, for the third case ω_n = 5rad/s, η = 0.05 and α = 0.8 are assumed; and finally, for the fourth case ω_n = 10rad/s, η = 0.05 and α = 0.2 are used. Accordingly, for all the cases from first to four, obtained fuzzy unit impulse responses are shown in Figures 7 to 10. By changing the parametric values as discussed for fuzzy step responses, one may observe that results obtained for this case show same behaviour as fuzzy step response.

In the case of unit impulse load, various numerical simulation results by changing the truncation order are also incorporated in Tables V and VI to show the convergence.

One may note the rapid convergence for different truncation order from Tables I to VI. Moreover, these tables clearly demonstrate that only a few numbers of approximations are sufficient for getting the convergence.

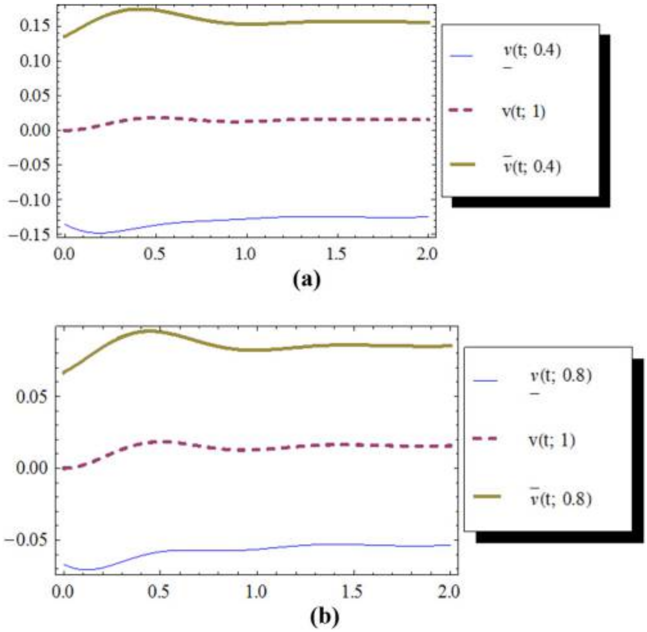

Figure 5.
Interval unit step response for (a) $r = 0.4$, (b) $r = 0.8$ with $\omega_n = 5$ rad/s, $\eta = 0.5$ $\alpha = 0.5$ and $r = 1$

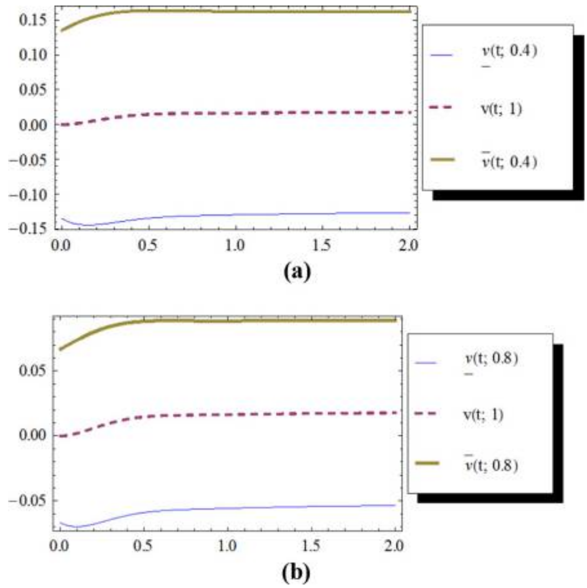

Figure 6.
Interval unit step response for (a) $r = 0.4$, (b) $r = 0.8$ with $\omega_n = 5$ rad/s, $\eta = 0.5$, $\alpha = 0.8$ and $r = 1$

No. of approximations	$\underline{v}(x,t,\alpha)$	$\overline{v}(x,t,\alpha)$
1	−0.2570579673	0.4296511612
2	−0.3037515074	0.3426942844
3	−0.2871350062	0.3622078397
4	−0.2899776321	0.3592397177
5	−0.2896880346	0.35953304
6	−0.2897079411	0.3595130517
7	−0.289706952	0.3595140422
8	−0.2897069892	0.359514005
9	−0.2897069881	0.3595140061
10	−0.2897069882	0.3595140061
11	−0.2897069882	0.3595140061
12	−0.2897069882	0.3595140061
13	−0.2897069882	0.3595140061
14	−0.2897069882	0.3595140061
15	−0.2897069882	0.3595140061
50	−0.2897069882	0.3595140061

Table I. Bounds of unit step response (for different number of approximations) for $\omega_n = 5,\ \eta = 0.05,\ x = 1/2,\ \alpha = 0.5,\ r = 0.1$ and $t = 0.6$

No. of approximations	$\underline{v}(x,t,\alpha)$	$\overline{v}(x,t,\alpha)$	Behera and Chakraverty (2013c) and Liang and Tang (2007)
1	0.08629659695	0.08629659695	0.08629659695
2	0.01947138847	0.01947138847	0.01947138847
3	0.03753641673	0.03753641673	0.03753641673
4	0.03463104279	0.03463104279	0.03463104279
5	0.03492250269	0.03492250269	0.03492250269
6	0.03490255529	0.03490255529	0.03490255529
7	0.03490354511	0.03490354511	0.03490354511
8	0.03490350788	0.03490350788	0.03490350788
9	0.03490350898	0.03490350898	0.03490350898
10	0.03490350895	0.03490350895	0.03490350895
11	0.03490350895	0.03490350895	0.03490350895
12	0.03490350895	0.03490350895	0.03490350895
13	0.03490350895	0.03490350895	0.03490350895
14	0.03490350895	0.03490350895	0.03490350895
15	0.03490350895	0.03490350895	0.03490350895
50	0.03490350895	0.03490350895	0.03490350895

Table II. Bounds of unit step response (for different number of approximations) for $\omega_n = 5,\ \eta = 0.05,\ x = 1/2,\ \alpha = 0.5,\ r = 1$ and $t = 0.6$

It is a gigantic task to include here all the results with respect to various parameters involved. For both the cases, for $r = 1$, fuzzy initial conditions convert into crisp initial conditions. It is interesting to note that for both the responses, lower and upper bounds of the fuzzy solutions are same for $r = 1$ And those are found to be same as Behera and Chakraverty (2013c) and Liang and Tang (2007).

Here, the initial condition has been taken as fuzzy with an idea that the condition may actually be uncertain, i.e. it may be because of error in observation or experiment, etc. where we have modelled the uncertainty in terms of Gaussian membership function. As such this will force the governing differential equation as a whole as uncertain. So, naturally the

Table III. Bounds of unit step response (for different number of approximations) for $\omega_n = 5$, $\eta = 0.5$, $x = 1/2$, $\alpha = 0.2$, $r = 0.4$ and $t = 0.5$

No. of approximations	$\underline{v}(x,t,\alpha)$	$\overline{v}(x,t,\alpha)$
1	−0.1431311166	0.2629875012
2	−0.1399348068	0.1735918129
3	−0.1440325514	0.1921654502
4	−0.1434076684	0.1898833849
5	−0.1434656363	0.1900574066
6	−0.1434620644	0.1900482501
7	−0.1434622225	0.1900486043
8	−0.1434622172	0.1900485937
9	−0.1434622174	0.190048594
10	−0.1434622174	0.190048594
11	−0.1434622174	0.190048594
12	−0.1434622174	0.190048594
13	−0.1434622174	0.190048594
14	−0.1434622174	0.190048594
15	−0.1434622174	0.190048594
50	−0.1434622174	0.190048594

Table IV. Bounds of unit step response (for different number of approximations) for $\omega_n = 5$, $\eta = 0.5$, $x = 1/2$, $\alpha = 0.2$, $r = 1$ and $t = 0.5$

No. of approximations	$\underline{v}(x,t,\alpha)$	$\overline{v}(x,t,\alpha)$	Behera and Chakraverty (2013c) and Liang and Tang (2007)
1	0.05992819233	0.05992810233	0.05992810233
2	0.01682850308	0.01682850308	0.01682850308
3	0.02406644936	0.02406644936	0.02406644936
4	0.02323785826	0.02323785826	0.02323785826
5	0.02329588519	0.02329588519	0.02329588519
6	0.02329309284	0.02329309284	0.02329309284
7	0.02329319086	0.02329319086	0.02329319086
8	0.02329318824	0.02329318824	0.02329318824
9	0.02329318829	0.02329318829	0.02329318829
10	0.02329318829	0.02329318829	0.02329318829
11	0.02329318829	0.02329318829	0.02329318829
12	0.02329318829	0.02329318829	0.02329318829
13	0.02329318829	0.02329318829	0.02329318829
14	0.02329318829	0.02329318829	0.02329318829
15	0.02329318829	0.02329318829	0.02329318829
50	0.02329318829	0.02329318829	0.02329318829

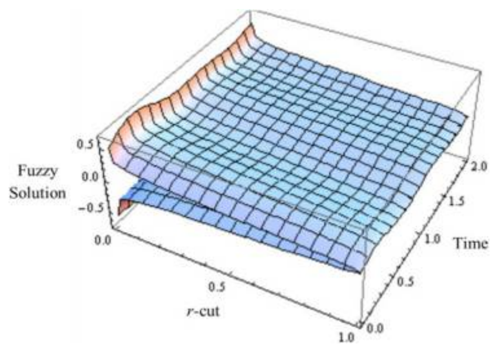

Figure 7. Fuzzy unit impulse response for $\omega_n = 5$ rad/s, $\eta = 0.5$ and $\alpha = 0.2$

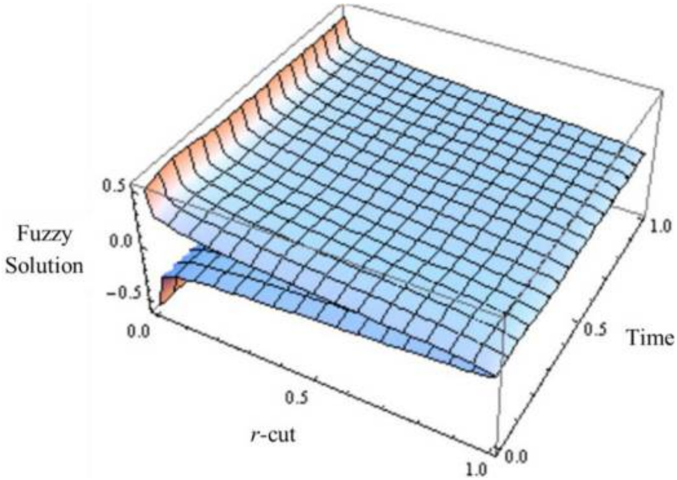

Figure 8.
Fuzzy unit impulse response for $\omega_n = 10$ rad/s, $\eta = 0.5$ and $\alpha = 0.5$

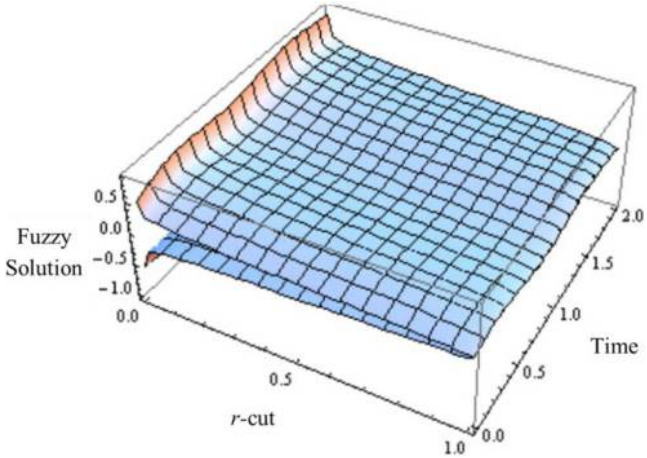

Figure 9.
Fuzzy unit impulse response for $\omega_n = 5$ rad/s, $\eta = 0.05$ and $\alpha = 0.8$

outcome or the output (result) must be uncertain. This way we may have the actual essence of the uncertainty in response which may benefit the engineers to understand the safety of the system in a better way.

7. Conclusions

VIM has successfully been applied to obtain the uncertain dynamic responses of fuzzy fractionally damped simply supported beam using double parametric form of fuzzy numbers. In general, we need to transform the fuzzy differential equation to two crisp differential equations. These differential equations may be coupled or uncoupled

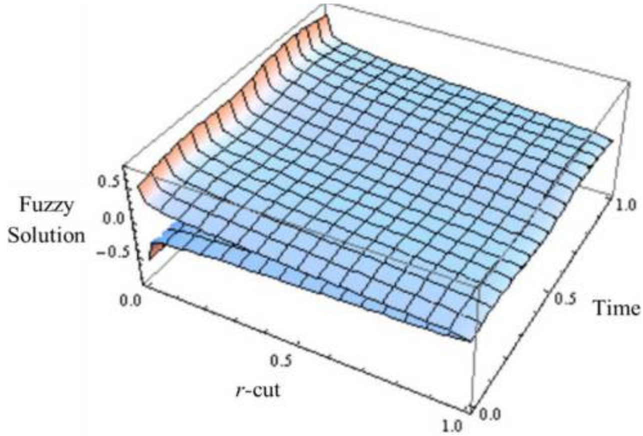

Figure 10. Fuzzy unit impulse response for $\omega_n = 10$ rad/s, $\eta = 0.05$ and $\alpha = 0.2$

Table V. Bounds of unit impulse response (for different number of approximations) for $\omega_n = 5$, $\eta = 0.5$, $x = 1/2$, $\lambda = 0.2$, $\alpha = 0.4$ and $t = 0.5$

No. of approximations	$\underline{v}(x,t,\alpha)$	$\overline{v}(x,t,\alpha)$
1	0.03665346039	0.4427720782
2	−0.2183478177	0.09517880195
3	−0.1546862099	0.1815117917
4	−0.1653795432	0.16791151
5	−0.1644116193	0.1691114236
6	−0.1644685594	0.1690417551
7	−0.1644662059	0.1690446209
8	−0.1644662781	0.1690445328
9	−0.1644662764	0.1690445349
10	−0.1644662764	0.1690445349
11	−0.1644662764	0.1690445349
12	−0.1644662764	0.1690445349
13	−0.1644662764	0.1690445349
14	−0.1644662764	0.1690445349
15	−0.1644662764	0.1690445349
50	−0.1644662764	0.1690445349

depending upon the sign of the coefficients. Accordingly, one has to solve the system of crisp differential equations. But it is interesting to note that the proposed double parametric based method does not require to solve the system of differential equations rather it only solves the transformed crisp differential equation. As such, it is very easy and straight forward to apply. Gaussian convex normalized fuzzy sets are considered for the present analysis. Uncertain dynamic responses subject to unit step and impulse loads are chosen to illustrate the proposed procedure. This method is found to be efficient for computing approximate solution bounds of uncertain differential equation for fractional order because only few terms are required for the convergence. However, in the present work, fuzziness or uncertainty are considered only in the involved initial conditions and accordingly uncertain responses have been computed. Hence, the future

Table VI. Bounds of unit impulse response (for different number of approximations) for $\omega_n = 5$, $\eta = 0.5$, $x = 1/2$, $\lambda = 0.2$, $\alpha = 1$ and $t = 0.5$

No. of approximations	$\underline{v}(x,t,\alpha)$	$\overline{v}(x,t,\alpha)$	Liang and Tang (2007)
1	0.2397127693	0.2397127693	0.2397127693
2	−0.06158450788	−0.06158450788	−0.06158450788
3	0.01341279089	0.01341279089	0.01341279089
4	0.00126598341	0.00126598341	0.00126598341
5	0.002349902194	0.002349902194	0.002349902194
6	0.00228659788	0.00228659788	0.00228659788
7	0.002289207517	0.002289207517	0.002289207517
8	0.002289127359	0.002289127359	0.002289127359
9	0.002289129269	0.002289129269	0.002289129269
10	0.002289129269	0.002289129269	0.002289129269
11	0.002289129269	0.002289129269	0.002289129269
12	0.002289129269	0.002289129269	0.002289129269
13	0.002289129269	0.002289129269	0.002289129269
14	0.002289129269	0.002289129269	0.002289129269
15	0.002289129269	0.002289129269	0.002289129269
50	0.002289129269	0.002289129269	0.002289129269

aim is to include fuzziness for all the associated parameters such as in material and geometric properties. And accordingly, the aim is to develop suitable numerical method to obtain the solution.

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