Reliability Engineering and System Safety 130 (2014) 175–189

Contents lists available at ScienceDirect

Reliability Engineering and System Safety

journal homepage: www.elsevier.com/locate/ress

Inverse Gaussian process models for degradation analysis: A Bayesian perspective

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ARTICLE INFO

Article history:
Received 27 October 2013
Received in revised form
3 June 2014
Accepted 5 June 2014
Available online 16 June 2014

Keywords:
Degradation model
Bayesian method
Inverse Gaussian process
Random effects

ABSTRACT

This paper conducts a Bayesian analysis of inverse Gaussian process models for degradation modeling and inference. Novel features of the Bayesian analysis are the natural manners for incorporating subjective information, pooling of random effects information among product population, and a straightforward way of coping with evolving data sets for on-line prediction. A general Bayesian framework is proposed for degradation analysis with inverse Gaussian process models. A simple inverse Gaussian process model and three inverse Gaussian process models with random effects are investigated using Bayesian method. In addition, a comprehensive sensitivity analysis of prior distributions and sample sizes is carried out through simulation. Finally, a classic example is presented to demonstrate the applicability of the Bayesian method for degradation analysis with the inverse Gaussian process models.

1. Introduction

Modern products evolve from generation to generation. It naturally gives rise to the continuing cutting down of time-to-market and the ever-increasing pace of new products appearing on the market. Meanwhile, product reliability has become an indispensable aspect of customer expectation. Increased reliability expectation with lower cost has become a critical issue for companies to deliver competitive products. Methods such as condition monitoring and degradation analysis are developed for reliability analysis of modern products. Degradation analysis is demonstrated as a significant toolkit, especially for the ones that are subjected to limited test time and sample size [1]. A comprehensive guide to degradation analysis is previously introduced by Meeker and Escobar [2]. Followed by a great amount of published papers, degradation related methods are introduced for various fields of reliability, which include reliability tests [3–5], reliability analysis [6–9], and fault prognostics [10–13]. Degradation modeling and parameter estimation are two indispensable aspects for the implementation of degradation analysis. A suitable degradation model is the key point for degradation characterization and reliability representation of a product. Meanwhile, a flexible estimation method is the key point for reliability assessment and degradation inference of a product. A high-precision reliability analysis of modern products consequently relies heavily on these two critical aspects of degradation analysis.

Considering the research on degradation modeling, the stochastic process based models are generally used [7,9]. Two most common classes of stochastic process are the gamma and the Wiener processes. These two classes have been well studied in the literature. The gamma process and its extensions in degradation modeling have been investigated in the works [14–16]. The applications of the Wiener process and its extensions in degradation modeling have also been investigated in the works [17,18]. Recently, the inverse Gaussian (IG) process has been reported as an attractive and flexible model for degradation modeling by Wang and Xu [19]. It has been demonstrated by them that the IG process model is more suitable than the Wiener and the gamma processes models for degradation modeling in some applications. Qin et al. [20] has also demonstrated the flexibility of IG process for degradation modeling through the application to the reliability analysis of energy pipelines. These two works were based on a simple IG process model. Ye and Chen [21] investigated the physical interpretation of IG process for degradation modeling and further introduced three IG process models with random effects by extending the simple IG model. An inverse normal-gamma mixture of an IG process model was also proposed by Peng [22]. These models were useful for the situations that random effects were associated with the degradation mean and variance of products. However, considering the situations that the random effects affect solely on the degradation mean, these models were limited due to the correlation between degradation mean and variance through random effects parameter. A classic example is the degradation data of

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http://dx.doi.org/10.1016/j.ress.2014.06.005
0951-8320/

Nomenclature

IG inverse Gaussian
MLE maximum likelihood estimation
EM expectation maximization
PDF probability density function
CDF cumulative distribution function
MCMC Markov chain Monte Carlo method
RD random drift IG process
RV random volatility IG process
RDV random drift-volatility IG process
$\boldsymbol{\theta}$ parameters of an IG process model
$\boldsymbol{\theta}^F$ parameters without random effects (fixed parameters)
$\boldsymbol{\theta}^R$ parameters with random effects (random parameters)
$\boldsymbol{\theta}^H$ hyper-parameters of probability distributions for the random parameters
$\pi(\boldsymbol{\theta})$ prior distribution
Y_i degradation data of the ith product with $i=1,\ldots,n$
$\boldsymbol{\theta}_i^R$ random parameters of the ith degradation process with $i=1,\ldots,n$
$\pi(\boldsymbol{\theta}_i^R|\boldsymbol{\theta}^H)$ prior distribution of random parameters $\boldsymbol{\theta}_i^R$ with hyper-parameters $\boldsymbol{\theta}^H$
Δy_{ij} degradation increment
$L(Y|\boldsymbol{\theta})$ likelihood function
$f(\Delta y_{ij}|\boldsymbol{\theta}^F)$ PDF of degradation increment under an IG process model without random effects
$f(\Delta y_{ij}|\boldsymbol{\theta}^F,\boldsymbol{\theta}_i^R)$ PDF of degradation increment under an IG process model with random effects
$p(\boldsymbol{\theta}^F,\boldsymbol{\theta}^R,\boldsymbol{\theta}^H|Y)$ posterior distribution
$Y_S(t)$ degradation process with a simple IG process model
$\Lambda(t)$ monotone increasing function in an IG process model
θ_Λ parameters of function $\Lambda(t)$
$\Phi(\cdot)$ CDF of a standard normal distribution
$\phi(\cdot)$ PDF of a standard normal distribution
$Pr(\cdot)$ probability of an event
$f(y|a,b)$ PDF of an IG distribution
$F(y|a,b)$ CDF of an IG distribution
$\mathrm{IG}(\mu\Delta\Lambda,\lambda\Delta\Lambda^2)$ IG process with function $\Lambda(t)$ and parameter μ and λ
$f_S(y|\mu\Lambda(t),\lambda\Lambda^2(t))$ PDF of a simple IG process model
$R_S(t|\mu\Lambda(t),\lambda\Lambda^2(t))$ reliability function of a product with a simple IG process model
$\mathrm{TN}(\omega,\kappa^{-2})$ truncated normal distribution with mean ω and variance κ^{-2}
$\mathrm{Uniform}(a,b)$ uniform distribution with boundary $[a,b]$
$\mathrm{Gamma}(\delta,\gamma)$ gamma distribution with shape parameter δ and rate parameter γ
Y_S degradation data with a simple IG process model
$L_S(Y_S|\mu,\lambda,\theta_\Lambda)$ likelihood function of Y_S with a simple IG process model
$p(\mu,\lambda,\theta_\Lambda|Y_S)$ posterior distribution of model parameters for a simple IG process model
$R_S(t|Y_S)$ inference of reliability for a product with a simple IG process model
$f_{S,m+1}(y|Y_S)$ prediction of degradation at t_{m+1} for a product with a simple IG process model
$Y_{RD}(t)$ degradation process with a RD model
$f_{RD}(y|\omega,\kappa,\Lambda(t),\lambda)$ PDF of a RD model
$R_{RD}(t|\omega,\kappa,\Lambda(t),\lambda)$ reliability function of a product with a RD model
Y_{RD} degradation data with a RD model
$\boldsymbol{\mu}$ set of random parameters
$l_{RD}(Y_{RD,i},\mu_i|\omega,\kappa,\theta_\Lambda,\lambda)$ likelihood contribution of the ith degradation path $Y_{RD,i}$ with a RD model
$L_{RD}(Y_{RD},\boldsymbol{\mu}|\omega,\kappa,\theta_\Lambda,\lambda)$ likelihood function of Y_{RD} and random parameters with a RD model
$p(\omega,\kappa,\theta_\Lambda,\lambda,\boldsymbol{\mu}|Y_{RD})$ posterior distribution of model parameters for a RD model
$R_{RD}(t|Y_{RD})$ inference of reliability for the product population with a RD model
$f_{RDi,m+1}(y|Y_{RD})$ prediction of degradation at $t_{i,m+1}$ for the ith product with a RD model
$Y_{RV}(t)$ degradation process with a RV model
$\Gamma(\bullet)$ gamma function
$f_{RV}(y|\mu,\Lambda(t),\delta,\gamma)$ PDF of a RV model
$R_{RV}(t|\mu,\Lambda(t),\delta,\gamma)$ reliability function of a product with a RV model
Y_{RV} degradation data with a RV model
$\boldsymbol{\lambda}$ set of random parameters
$L_{RV}(Y_{RV},\boldsymbol{\lambda}|\mu,\theta_\Lambda,\delta,\gamma)$ likelihood function of Y_{RV} and random parameters with a RV model
$p(\mu,\theta_\Lambda,\delta,\gamma,\boldsymbol{\lambda}|Y_{RV})$ posterior distribution of model parameters for a RV model
$R_{RV}(t|Y_{RV})$ inference of reliability for the product population with a RV model
$f_{RVi,m+1}(y|Y_{RV})$ prediction of degradation at $t_{i,m+1}$ for the ith product with a RV model
$Y_{RDV}(t)$ degradation process with a RDV model
$f_{RDV}(y|\omega,\kappa,\Lambda(t),\lambda)$ PDF of a RDV model
$R_{RDV}(t|\omega,\kappa,\Lambda(t),\lambda)$ reliability function of a product with a RDV model
Y_{RDV} degradation data with a RDV model
$L_{RDV}(Y_{RDV},\boldsymbol{\mu}|\omega,\kappa,\theta_\Lambda,\lambda)$ likelihood function of Y_{RDV} and random parameters with a RDV model
$p(\omega,\kappa,\theta_\Lambda,\lambda,\boldsymbol{\mu}|Y_{RDV})$ posterior distribution of model parameters for a RDV model
Y_A degradation data with subscript A representing a specific IG process model
$F_A(y|\theta_A)$ CDF of an IG process model
$S(\tilde{\theta}_A)$ Bayesian χ^2 test statistic
B_p Bayesian χ^2 test probability

the GaAs Laser investigated by Wang and Xu [19] and Ye and Chen [21]. Unit-specific heterogeneity of degradation rate among product population is significant, yet the variance of degradation increments within a specific unit is small. A degradation model with random effects affect solely on the degradation mean is needed for the degradation modeling of this GaAs Laser degradation data. Accordingly, the IG process for degradation modeling still deserves further investigation to make it more versatile for various situations of degradation data.

Considering the research on parameter estimation, the maximum likelihood estimation (MLE) is often the tool of choice to implement parameter estimation for the IG process models. Wang and Xu [19], Ye and Chen [21] and Peng [22] have introduced the MLE for the IG process model using expectation maximization (EM) and bootstrap methods. Nowadays, two typical situations are generally encountered in degradation analysis of modern product, i.e. (1) the degradation analysis with sparse/fragmented degradation observations, and (2) the degradation analysis with

evolving/updating degradation observations. The first situation is commonly introduced by the reliability analysis of the products that cannot be monitored frequently, such as the underground oil and natural gas pipelines [20]. Subjective information or historical information is generally incorporated to complement the insufficiency of these sparse/fragmented degradation observations [23,24]. In addition, it is hard for the MLE-based method to carry out the degradation analysis under this situation. A degradation analysis method for information integration is needed. The second situation is generally introduced by the system health management of the products that are subject to condition monitoring, such as the super luminescent diode [9] and the GaAs Laser [19]. The degradation analysis results are updated when newly observed degradation data is available [25,26]. A degradation analysis method for model updating is needed as well. For the degradation analysis with subjective information and continual monitoring data, Bayesian method has become a standard toolkit [3,9,11]. However, little work has been published on the degradation analysis with IG process using Bayesian method. Rare exceptions are the works presented by Qin et al. [20] and Peng et al. [27]. Qin et al. [20] presented the parameter estimation of a simple IG process model using Bayesian method. Their method was limited to the simple IG process model. Moreover, the aspects concerning hierarchical priors for random effects information fusion and posterior analysis for degradation analysis results updating were not well studied. In our previous work [27], the Bayesian method for degradation analysis with a random drifts IG process model introduced by Ye and Chen [21] was presented. A Bayesian χ^2 goodness-of-fit test was introduced for this random drifts IG process model. However, this work was limited to a specific IG process model and various aspects were not well developed. An improvement of the random drifts IG process model and a further extension of the proposed Bayesian method for more general situation is needed.

The objective of this paper is to present a comprehensive Bayesian analysis of inverse Gaussian process models for degradation modeling and inference. In comparison with the results reported in [19–22] and our previous work [27], the following aspects are addressed in this paper: (1) a general Bayesian framework for degradation analysis with IG process models is constructed, where four IG process models are derived progressively under the Bayesian framework, (2) a natural manner for subjective information integrating is introduced, where sensitivity analysis for prior distributions and sample sizes is studied through simulation, (3) a hierarchical Bayesian method for random effect information fusion is presented, where there IG process models with random effects are investigated using hierarchical Bayesian method and the random drift IG process model introduced by Ye and Chen [21] is improved, and (4) a straightforward way of coping with evolving degradation observations is introduced, where a classical example is incorporated to demonstration the applicability of the proposed Bayesian method. Finally, the effectiveness of the proposed Bayesian framework is demonstrated through numerical comparisons. The degradation analysis results of an illustrative example using the proposed Bayesian method are compared with the ones obtained using MLE-based method introduced by Ye and Chen [21].

The outline of the paper is as follows. A general Bayesian framework for the IG process model is constructed in Section 2 with specific descriptions of three critical aspects. The Bayesian framework is then implemented to the IG process models in Section 3. The derivation of hierarchical priors, the formulization of posterior distribution, and the implementation of posterior analysis are described in this section. A Bayesian χ^2 goodness-of-fit test for the IG process models is introduced in Section 4. Section 5 describes the simulation study of sensitivity analysis for the Bayesian analysis subjected to different sample sizes and choices of prior distributions. In Section 6, the presented Bayesian analysis for three IG process models with random effects is then applied to the GaAs Lasers data. We then conclude the paper in Section 7 with a brief description of possible topics for future research.

2. A general Bayesian framework for the IG process models

When the degradation process of a product is observed and an IG process degradation model is chosen for degradation modeling, the associated parameter estimation, reliability inference and degradation prediction are indispensable aspects of degradation analysis. This section aims to develop a coherent Bayesian framework for degradation analysis with IG process models. Aiming at the IG process models in this paper, a general Bayesian framework is first constructed and depicted in Figs. 1–3.

As depicted in Fig. 1, $\boldsymbol{\theta}$ is a parameter set for a specific IG process model and a joint prior distribution $\pi(\boldsymbol{\theta})$ is used to describe prior information about these parameters. Since there are different types of unknown parameters, we use $\boldsymbol{\theta}^F$, $\boldsymbol{\theta}^R$ and $\boldsymbol{\theta}^H$ with $\boldsymbol{\theta}=\{\boldsymbol{\theta}^F,\boldsymbol{\theta}^R,\boldsymbol{\theta}^H\}$ separately denoting the parameters without random effects (fixed parameters), the parameters with random effects (random parameters), and the hyper-parameters of probability distributions for the random parameters. The prior distributions for fixed parameters are derived based on the available information, such as subjective information from experts' judgment and historical information from similar products. The prior distributions for random parameters are derived using hierarchical priors. It aims to pool random effects information among product population. For each degradation path Y_i with $i=1,\ldots,n$, a prior distribution $\pi(\boldsymbol{\theta}_i^R|\boldsymbol{\theta}^H)$ is specified for $\boldsymbol{\theta}_i^R$. It is used to model the unit-to-unit variability among the product population. Simultaneously, hierarchical prior distribution $\pi(\boldsymbol{\theta}^H)$ is specified for $\boldsymbol{\theta}^H$ which are hyper-parameters of the prior distributions$\pi(\boldsymbol{\theta}_i^R|\boldsymbol{\theta}^H)$. These priors are used jointly to integrate random effects information among different degradation paths. Specific descriptions of prior

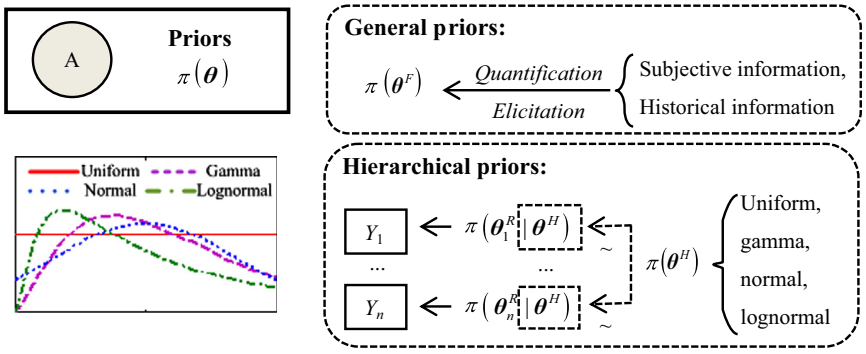

Fig. 1. A general Bayesian framework for the IG process models: elicitation of hierarchical prior distributions.

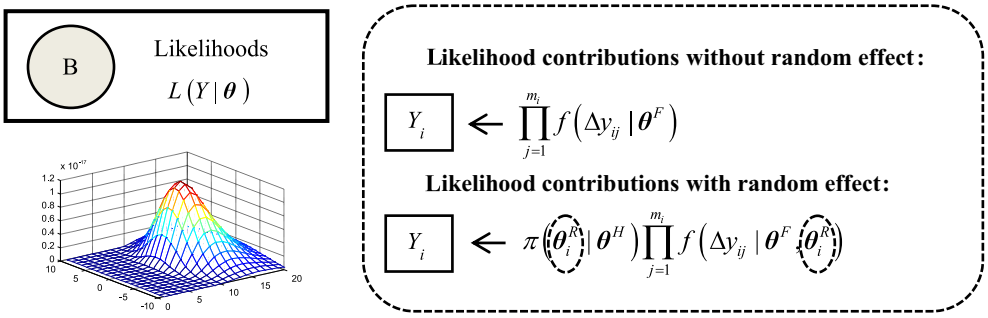

Fig. 2. A general Bayesian framework for the IG process models: construction of likelihood function.

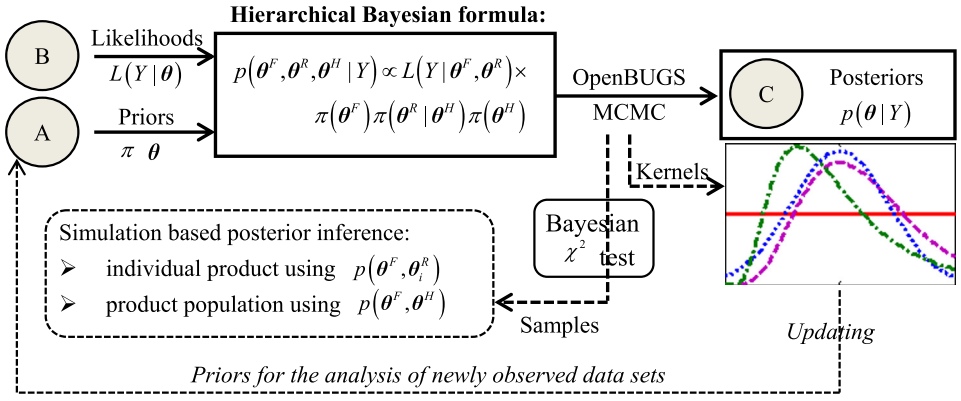

Fig. 3. A general Bayesian framework for the IG process models: formulation of posterior distribution and implementation of posterior analysis.

elicitation and construction of hierarchical priors for the IG process models are separately described in Sections 3.1 and 3.2.

Then the Bayesian framework moves to Fig. 2. The information of degradation data Y are delivered through the construction of the likelihood function $L(Y|\boldsymbol{\theta})$. It is constructed by multiplying relevant likelihood contributions of degradation paths Y_i under specific IG process models. For the IG process model without random effects, the likelihood contribution is formulized by multiplying the probability density functions (PDF) of degradation increments Δy_{ij} with $j=1,\ldots,m_i$, which is given in Fig. 2 as $f(\Delta y_{ij}|\boldsymbol{\theta}^F)$. For the IG process model with random effects, the likelihood contribution is formulized by multiplying the PDF of degradation increments Δy_{ij}, as well as the PDF of random parameter $\boldsymbol{\theta}_i^R$, which is given in Fig. 2 as $f(\Delta y_{ij}|\boldsymbol{\theta}^F)$ and $\pi(\boldsymbol{\theta}_i^R|\boldsymbol{\theta}^H)$. Detailed constructions of likelihood functions for the IG process models are separately described in Section 3.

After prior distribution and likelihood function are separately derived in Figs. 1 and 2, the Bayesian framework then moves to Fig. 3. Since random effects are incorporated using hierarchical priors, the hierarchical Bayesian formula is used to formulate the posterior distribution $p(\boldsymbol{\theta}^F,\boldsymbol{\theta}^R,\boldsymbol{\theta}^H|Y)$. It is a joint posterior distribution of parameters with different types as $\boldsymbol{\theta}^F$, $\boldsymbol{\theta}^R$ and $\boldsymbol{\theta}^H$. To facilitate posterior analysis, the Markov chain Monte Carlo method (MCMC) is used to generate samples from the posterior distribution. It is implemented through the software package OpenBUGS [28]. The generate samples are then tested using a Bayesian goodness-of-fit test. Finally, simulation based posterior inferences are carried out with these validated posterior samples. For the posterior inference of individual product, the posterior samples of parameters $\boldsymbol{\theta}^F$ and $\boldsymbol{\theta}_i^R$ are used. For the posterior inference of product population, the posterior samples of parameters $\boldsymbol{\theta}^F$ and $\boldsymbol{\theta}^H$ are used. In addition, the kernel distributions obtained from these posterior samples are further incorporated as prior distributions for the analysis of newly observed data. It is aimed to facilitate the implementation of the proposed Bayesian method in the scenario of condition monitoring, where on-line updating of degradation analysis is required. Specific derivation of posterior distribution and description of posterior analysis for different IG process models are presented in Section 3. A Bayesian χ^2 goodness-of-fit test for the IG process models is developed in Section 4.

To implement this Bayesian framework on the IG process models, three aspects are highlighted in Section 3, (1) the derivation and selection of prior distributions, (2) the formulization of likelihood functions, and (3) the obtaining of posterior distributions along with code scripts in OpenBUGS and the associated formulations for posterior analysis.

3. Bayesian analysis for the IG process models

3.1. Bayesian analysis for a simple IG process model

3.1.1. Model description

A simple IG process model with function $\Lambda(t)$ and parameters μ and λ is defined for a degradation process $\{Y_S(t), t\geq 0\}$ with $Y_S(0)\equiv 0$. It has the following properties [19,21]:

$Y_S(t)$ has independent increments, i.e., $Y_S(t_4)-Y_S(t_3)$ and $Y_S(t_2)-Y_S(t_1)$ are independent for $\forall t_4>t_3\geq t_2>t_1$, and

the degradation increment $Y_S(t+\Delta t)-Y_S(t)$ follows an IG distribution as $\mathrm{IG}(\mu\Delta\Lambda,\lambda\Delta\Lambda^2)$,

where $\Delta\Lambda=\Lambda(t+\Delta t)-\Lambda(t)$ and $\Lambda(t)$ is a monotone increasing function. In this paper, to present a general Bayesian framework for the IG process models, we use a general function $\Lambda(t)$ with parameter θ_Λ in the following sections. Various particular forms of $\Lambda(t)$ can be used according to specific engineering applications, such as a power-law function, an exponential function, and even a physical-model based function. The Bayesian analysis presented below can also be applied to the IG process models with specific forms of $\Lambda(t)$ by substituting the particular form of $\Lambda(t)$ and assigning prior distributions for parameters θ_Λ. A further illustration with $\Lambda(t)=t^q$ is

presented in the simulation study section and the illustrative example section as well.

The PDF of the IG distribution for $y \sim \mathrm{IG}(a,b), a,b>0$ with mean a and variance a^3/b is

$$f(y|a,b)=\sqrt{\frac{b}{2\pi y^3}}\exp\left[-\frac{b(y-a)^2}{2a^2y}\right], \quad y>0 \tag{1}$$

and its cumulative distribution function (CDF) is

$$F(y|a,b)=\Phi\left[\sqrt{\frac{b}{y}}\left(\frac{y}{a}-1\right)\right]+\exp\left(\frac{2b}{a}\right)\Phi\left[-\sqrt{\frac{b}{y}}\left(\frac{y}{a}+1\right)\right] \tag{2}$$

where $\Phi(\cdot)$ is the standard normal CDF.

The degradation process is then described as $Y_S(t)\sim \mathrm{IG}(\mu\Lambda(t), \lambda\Lambda^2(t)), \mu,\lambda>0$. The PDF of the simple IG process model for $Y_S(t)$ is obtained as

$$f_S(y|\mu\Lambda(t),\lambda\Lambda^2(t))=\sqrt{\frac{\lambda\Lambda^2(t)}{2\pi y^3}}\exp\left[-\frac{\lambda(y-\mu\Lambda(t))^2}{2\mu^2 y}\right] \tag{3}$$

Generally, a component with a degradation process $Y_S(t)$ fails when its degradation path firstly reaches a predefined threshold D. The reliability function of this component is obtained by considering the monotonicity property of the IG process and the distribution characteristics of degradation increments as

$$\begin{aligned} R_S(t|\mu\Lambda(t),\lambda\Lambda^2(t)) &= Pr(Y_S(t)-Y_S(0)<D|\mu,\Lambda(t),\lambda) \\ &= \Phi\left[\sqrt{\frac{\lambda}{D}}\left(\frac{D}{\mu}-\Lambda(t)\right)\right]+\exp\left(\frac{2\lambda\Lambda(t)}{\mu}\right)\Phi\left[-\sqrt{\frac{\lambda}{D}}\left(\frac{D}{\mu}+\Lambda(t)\right)\right] \end{aligned} \tag{4}$$

A pictorial description of degradation paths for the simple IG process model is presented in Fig. 4. Intuitively, since no random effects is included in this model, all the parameters μ, $\Lambda(t)$ and λ are unknown parameters. The degradation paths and the induced lifetime distribution have relevantly narrow spreads. This simple IG process model is suitable for the degradation modeling of a product for which no significant unit-to-unit variation within a population has been observed.

3.1.2. Bayesian analysis

Suppose the degradation observations of $Y_S(t)$ are observed for n units. Let $Y_S(t_{ij})$ denote the jth observation for unit i at time point t_{ij} with $j=1,\ldots,m_i$ and $i=1,\ldots,n$. Let $\Delta y_{ij}=Y_S(t_{ij})-Y_S(t_{i,j-1})$ be the degradation increment with $Y_S(t_{i,0})=0$. Under the simple IG process model, the Δy_{ij}s are independent and follow the IG distribution $\mathrm{IG}(\mu\Delta\Lambda_{ij},\lambda\Delta\Lambda_{ij}^2)$. Following the framework presented in Section 2, the Bayesian analysis for the simple IG process model is presented progressively as follow.

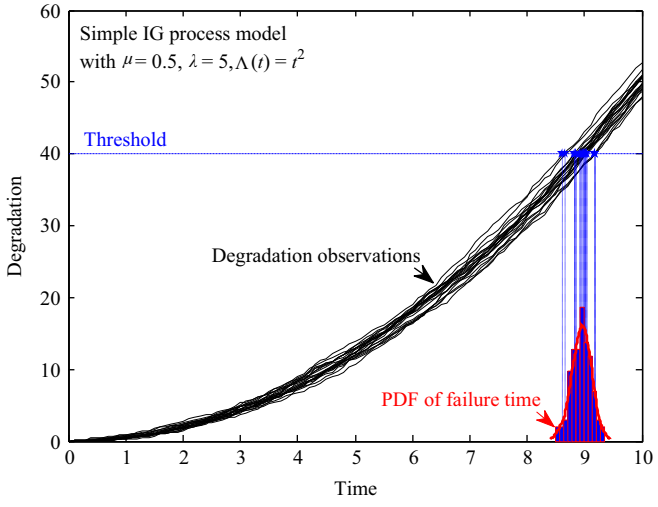

Fig. 4. A simulated simple IG process model with threshold $D=40$.

Following the framework presented in Fig. 1, suppose prior distributions for unknown parameters μ, λ, and θ_Λ for $\Lambda(t)$ are specified based on prior information. Since no random effects is considered for these parameters, non-hierarchical prior distributions are ascribed for these parameters as

$$\mu\sim \mathrm{TN}(\omega,\kappa^{-2}), \quad \theta_\Lambda\sim \mathrm{Uniform}(a_\Lambda,b_\Lambda), \quad \lambda\sim \mathrm{Gamma}(\delta,\gamma) \tag{5}$$

where $\mathrm{TN}(\omega,\kappa^{-2})$ is a truncated normal distribution with mean ω and variance κ^{-2}, $\mathrm{Uniform}(a_\Lambda,b_\Lambda)$ is a uniform distribution with boundary $[a_\Lambda,b_\Lambda]$, and $\mathrm{Gamma}(\delta,\gamma)$ is a gamma distribution with shape parameter δ and rate parameter γ.

The prior distributions for μ and λ are chosen based on the considerations that when random effects is introduced for these parameters, the truncated normal distribution and gamma distribution are separately chosen as the characteristic distributions for random effects modeling [21]. The choosing of random effects distributions can provide a certain tendency for the choice of prior distributions. Moreover, the truncated normal distribution is easy to be handled with hyper-parameters ω and κ^{-2} as mean and variance in quantification of subjective information. This is because that the parameter μ is related to the degradation rate in a degradation curve. Expert testimony about this degradation rate can be directly specified on its hyper-parameters (Section 5.3 in [29]). Similarly, the gamma distribution also possesses explicit parameters for experts' probabilities elicitation. On the other hand, direct information is hardly to be acquired for parameter θ_Λ. A general non-informative prior in the form of uniform distribution is chosen (Section 2.6 in [30]) for θ_Λ. Prior derivation is critical for Bayesian analysis, especially when field data is scarce. For more information about subjective information quantification and prior distribution derivation please refer to the literature [31–34].

For degradation observation Y_S, Δy_{ij}s are independent and follow IG distribution $\mathrm{IG}(\mu\Delta\Lambda_{ij},\lambda\Delta\Lambda_{ij}^2)$. The likelihood function for this degradation data under simple IG process model is obtained as

$$L_S(Y_S|\mu,\lambda,\theta_\Lambda)=\prod_{i=1}^{n}\prod_{j=1}^{m_i} f(\Delta y_{ij}|\mu\Delta\Lambda_{ij},\lambda\Delta\Lambda_{ij}^2) \tag{6}$$

where $\prod_{j=1}^{m_i} f(\Delta y_{ij}|\mu\Lambda_{ij},\lambda\Lambda_{ij}^2)$ is the likelihood contribution of the ith degradation paths $Y_{S,i}$. It is presented in Fig. 2 with $f(\Delta y_{ij}|\mu\Delta\Lambda_{ij},\lambda\Delta\Lambda_{ij}^2)$ given as $f(\Delta y_{ij}|\boldsymbol{\theta}^F)$.

Based on the prior distributions and likelihood function above, the joint posterior distribution for fixed parameters μ, λ and θ_Λ are obtained as

$$\begin{aligned} p(\mu,\lambda,\theta_\Lambda|Y_S) &\propto \pi(\mu)\pi(\lambda)\pi(\theta_\Lambda)L_S(Y_S|\mu,\lambda,\theta_\Lambda) \\ &\propto \phi[\kappa(\mu-\omega)]\lambda^{\delta-1}\exp(-\gamma\lambda)\prod_{i=1}^{n}\prod_{j=1}^{m_i}\sqrt{\lambda\Delta\Lambda_{ij}^2}\exp\left[-\frac{\lambda(\Delta y_{ij}-\mu\Delta\Lambda_{ij})^2}{2\mu^2\Delta y_{ij}}\right] \end{aligned} \tag{7}$$

where $\phi(\cdot)$ is the standard normal PDF, and the posterior distribution is given in the form of probabilistic kernel, where the proportionality constant part that does not related to the model parameter are omitted.

Obviously, there is no analytical expression for this joint posterior distribution. However, the MCMC can be utilized to generate samples from this joint posterior distribution. Moreover, a well-developed software package, OpenBUGS, is used to carry out the implementation of the MCMC. For details regarding MCMC and BUGS, readers are referred to the works [35,36] for construction of the MCMC algorithm, and the books [37,38] for coding within R and BUGS.

Based on the joint posterior distribution, the inference of reliability function and the prediction of degradation at time point t_{m+1} can be obtained as

$$R_S(t|Y_S)=\int_{\mu,\lambda,\theta_\Lambda} R_S(t|\mu\Lambda(t),\lambda\Lambda^2(t))p(\mu,\lambda,\theta_\Lambda|Y_S)d\mu d\lambda d\theta_\Lambda \tag{8}$$

$$f_{S,m+1}(y|Y_S)=\int_{\mu,\lambda,\theta_\Lambda} f_S(y|\mu\Lambda(t_{m+1}),\lambda\Lambda^2(t_{m+1}))p(\mu,\lambda,\theta_\Lambda|Y_S)d\mu d\lambda d\theta_\Lambda \tag{9}$$

where $f_{S,m+1}(y|Y_S)$ is the predicted PDF of degradation at time point t_{m+1}, and $f_S(y|\mu\Lambda(t),\lambda\Lambda^2(t))$ is the PDF of degradation process $Y_S(t)$ given in Eq. (3).

The calculations of Eqs. (8) and (9) are implemented through simulation based integration. By calculating the relevant values of $R_S(t|\tilde{\mu}\tilde{\lambda}(t),\tilde{\lambda}\tilde{\lambda}^2(t))$ and $f_{S,m+1}(y|\tilde{\mu}\tilde{\lambda}(t_{m+1}),\tilde{\lambda}\tilde{\lambda}^2(t_{m+1}))$ at each generated posterior sample $(\tilde{\mu},\tilde{q},\tilde{\lambda})$ from $p(\mu,\lambda,\theta_\Lambda|Y_S)$, statistical summarizations such as mean, variance, and kernel density distribution for $R_S(t|Y_S)$ and $f_{S,m+1}(y|Y_S)$ are obtained.

3.2. Bayesian analysis for a random drift IG process model

3.2.1. Model description

To account for the heterogeneity within a product population, random effects are introduced in the simple IG process model to generate random effects models. Ye and Chen [21] introduced a random drift IG process (RD) model by letting μ in the simple IG model follow a truncated normal distribution. Strictly speaking, it is not a real RD model since both the mean and variance of the degradation process are affected by the random parameter μ. In this paper, an improvement of the RD model is introduced by modifying the parameter structure of the simple IG process model into $Y(t)\sim \mathrm{IG}(\mu\Lambda(t),\mu^3\Lambda^2(t)/\lambda)$. The degradation mean of this model is $\mu\Lambda(t)$ and the variance is $\lambda\Lambda(t)$. By letting μ follow a truncated normal distribution, only the degradation mean is affected by the random parameter. It then gives rise to the RD model in this paper as $Y_{RD}(t)\sim \mathrm{IG}(\mu\Lambda(t),\mu^3\Lambda^2(t)/\lambda)$, $\mu\sim \mathrm{TN}(\omega,\kappa^{-2})$, $\lambda>0$, where the PDF of the truncated normal distribution is

$$g_\mu(\mu|\omega,\kappa^{-2})=\frac{\kappa\phi[\kappa(\mu-\omega)]}{1-\Phi(-\kappa\omega)},\quad \mu>0,\kappa>0 \tag{10}$$

The PDF of degradation process $Y_{RD}(t)$ for the RD model is obtained by averaging the PDF of a simple IG process model with modified parameter structure $\mathrm{IG}(\mu\Lambda(t),\mu^3\Lambda^2(t)/\lambda)$ in the whole range of μ as

$$\begin{aligned} f_{RD}(y|\omega,\kappa,\Lambda(t),\lambda)&=\int_{\mu>0} f_S(y|\mu\Lambda(t),\mu^3\Lambda^2(t)/\lambda)g_\mu(\mu|\omega,\kappa^{-2})d\mu \\ &=\int_{\mu>0}\sqrt{\frac{\mu^3\Lambda^2(t)}{2\pi\lambda y^3}}\exp\left[-\frac{\mu(y-\mu\Lambda(t))^2}{2\lambda y}\right]\frac{\kappa\phi[\kappa(\mu-\omega)]}{1-\Phi(-\kappa\omega)}d\mu \end{aligned} \tag{11}$$

and its reliability function with failure threshold D is obtained in the same way as

$$\begin{aligned} R_{RD}(t|\omega,\kappa,\Lambda(t),\lambda)&=\int_{\mu>0} R_S(t|\mu\Lambda(t),\mu^3\Lambda^2(t)/\lambda)g_\mu(\mu|\omega,\kappa^{-2})d\mu \\ &=\int_{\mu>0}\left\{\Phi\left[\sqrt{\frac{\mu}{\lambda D}}(D-\mu\Lambda(t))\right]+\exp\left(\frac{2\mu^2\Lambda(t)}{\lambda}\right)\right. \\ &\left.\times\Phi\left[-\sqrt{\frac{\mu}{\lambda D}}(D+\mu\Lambda(t))\right]\right\}\frac{\kappa\phi[\kappa(\mu-\omega)]}{1-\Phi(-\kappa\omega)}d\mu \end{aligned} \tag{12}$$

A pictorial description of degradation paths for this RD model is presented in Fig. 5. Since random effects are considered in the parameter μ, the mean function of the IG process $\mu\Lambda(t)$ varies in a certain extent. It leads to the dispersion of the degradation rate and the larger variance of the induced lifetime distribution. This model is appropriate for the degradation modeling of products for which significant variation of degradation rate is observed within the product population.

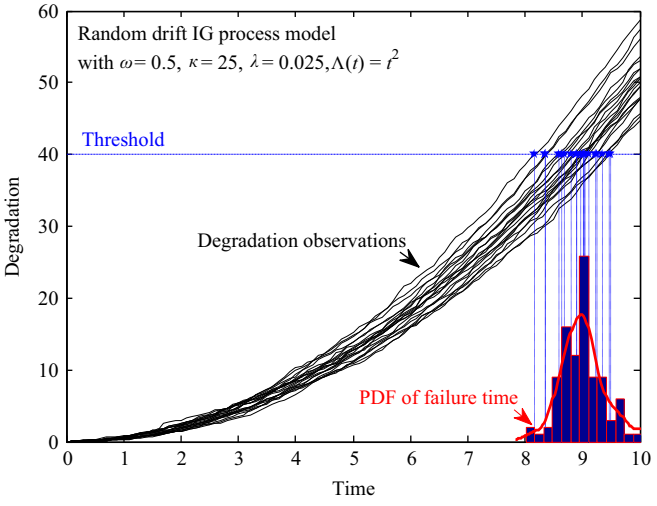

Fig. 5. A simulated RD model with threshold $D=40$.

3.2.2. Bayesian analysis

Given the degradation processes $Y_{RD}(t)$ at some discrete observation times for n units are observed as Y_{RD}. Then under the RD IG process model, the degradation increments $\Delta y_{ij}=Y_{RD}(t_{ij})-Y_{RD}(t_{i,j-1})$ are independent and follow IG distribution $\mathrm{IG}(\mu_i\Delta\Lambda_{ij},\mu_i^3\Delta\Lambda_{ij}^2/\lambda)$, $\mu_i\sim \mathrm{TN}(\omega,\kappa^{-2})$, $\Delta\Lambda_{ij}=\Lambda(t_{ij})-\Lambda(t_{i,j-1})$.

According to the framework presented in Fig. 1, prior distributions for the RD model are given as

$$\begin{cases}\omega\sim \mathrm{TN}(a_\omega,b_\omega^{-2})\\ \kappa\sim \mathrm{TN}(a_\kappa,b_\kappa^{-2})\end{cases},\quad \begin{cases}\theta_\Lambda\sim \mathrm{Uniform}(a_\Lambda,b_\Lambda)\\ \lambda\sim \mathrm{Gamma}(\delta,\gamma)\end{cases} \tag{13}$$

Since random effects are incorporated in parameter μ, each degradation path possesses a specific parameter μ_i that follows $\mathrm{TN}(\omega,\kappa^{-2})$ with $i=1,\ldots,n$. To pool random effects information among different degradation paths, we let all μ_i follow the same prior distributions with hyper-parameters ω and κ. The prior distributions for these hyper-parameters are specified as in Eq. (13). Compared with the prior distributions for the simple IG process model given in Eq. (5), the prior distributions specified for the RD model are delivered in a hierarchical way. It can also be seen from Fig. 1 that each degradation path Y_i possesses a specific parameter $\boldsymbol{\theta}_i^R\sim\pi(\boldsymbol{\theta}_i^R|\boldsymbol{\theta}^H)$ with all the hyper-parameters $\boldsymbol{\theta}^H$ specified with the same prior distributions. Based on this type of prior distributions, the estimations of parameters $\boldsymbol{\theta}^H$ depend on observation data from all the degradation paths. Accordingly, the inference of unit-specific parameter $\boldsymbol{\theta}_i^R$ benefits from the information provided by other degradation paths.

For observed degradation data Y_{RD}, the likelihood contribution of the ith degradation path $Y_{RD,i}$ with degradation increments Δy_{ij} and random parameter μ_i is obtained as follow

$$l_{RD}(Y_{RD,i},\mu_i|\omega,\kappa,\theta_\Lambda,\lambda)=g_\mu(\mu_i|\omega,\kappa^{-2})\prod_{j=1}^{m_i} f(\Delta y_{ij}|\mu\Delta\Lambda(t),\mu^3\Delta\Lambda^2(t)/\lambda) \tag{14}$$

This likelihood contribution is presented in Fig. 2 with $f\left(\Delta y_{ij}|\mu\Delta\Lambda(t),\mu^3\Delta\Lambda^2(t)/\lambda\right)$ given as $f(\Delta y_{ij}|\boldsymbol{\theta}^F,\boldsymbol{\theta}_i^R)$, and $g_\mu(\mu_i|\omega,\kappa^{-2})$ given as $\pi(\boldsymbol{\theta}_i^R|\boldsymbol{\theta}^H)$.

Then the likelihood function for the observed degradation data Y_{RD} is obtained as follow

$$L_{RD}(Y_{RD},\boldsymbol{\mu}|\omega,\kappa,\theta_\Lambda,\lambda)=\prod_{i=1}^{n} l_{RD}(Y_{RD,i},\mu_i|\omega,\kappa,\theta_\Lambda,\lambda) \tag{15}$$

where $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_n)$ includes all the random parameters for the degradation paths Y_{RD} and all μ_i follow the sample distribution function with the same parameters ω and κ.

As done in Section 3.1.2, the joint posterior distribution for both fixed parameters ω, κ, θ_Λ and λ, and random parameters $\boldsymbol{\mu}$ is obtained as

$$\begin{aligned}p(\omega,\kappa,\theta_\Lambda,\lambda,\boldsymbol{\mu}|Y_{RD}) &\propto \pi(\omega)\pi(\kappa)\pi(\theta_\Lambda)\pi(\lambda)L_{RD}(Y_{RD},\boldsymbol{\mu}|\omega,\kappa,\theta_\Lambda,\lambda)\\ &\propto \phi\left[b_\omega(\omega-a_\omega)\right]\phi\left[b_\kappa(\kappa-a_\kappa)\right]\lambda^{\delta-1}\exp(-\gamma\lambda)\times\\ &\prod_{i=1}^{n}\left\{\frac{\kappa\phi\left[\kappa(\mu_i-\omega)\right]}{1-\Phi(-\kappa\omega)}\prod_{j=1}^{m_i}\sqrt{\frac{\mu_i^3\Delta\Lambda_{ij}^2}{\lambda}}\exp\left[-\frac{\mu_i(\Delta y_{ij}-\mu_i\Delta\Lambda_{ij})^2}{2\lambda\Delta y_{ij}}\right]\right\}\end{aligned} \tag{16}$$

Based on the joint posterior distribution, the inference of reliability for the product population and the prediction of degradation for the ith product are obtained as

$$R_{RD}(t|Y_{RD})=\int_{\omega,\kappa,\theta_\Lambda,\lambda} R_{RD}(t|\omega,\kappa,\Lambda(t),\lambda)p(\omega,\kappa,\theta_\Lambda,\lambda|Y_{RD})d\omega d\kappa d\theta_\Lambda d\lambda \tag{17}$$

$$f_{RDi,m+1}(y|Y_{RD})=\int_{\mu_i,\theta_\Lambda,\lambda} f_S(y|\mu_i\Lambda(t),\mu_i^3\Lambda^2(t)/\lambda)p(\mu_i,\theta_\Lambda,\lambda|Y_{RD})d\mu_i d\theta_\Lambda d\lambda \tag{18}$$

where $f_{RDi,m+1}(y|Y_{RD})$ is the predicted PDF of degradation for the ith unit at time point $t_{i,m+1}$, and $p(\omega,\kappa,\theta_\Lambda,\lambda|Y_{RD})$ and $p(\mu_i,\theta_\Lambda,\lambda|Y_{RD})$ are the marginal joint posterior distributions for the corresponding parameters, which are obtained from the joint posterior distribution given in Eq. (16) by integrating over the parameters being excluded.

The calculations of Eq. (16) are implemented using the MCMC method through OpenBUGS. The code script for the RD model is attached in the Appendix. Statistical summaries of Eqs. (17) and (18) are obtained using a simulation based integration method based on the posterior samples of Eq. (16), which are similar to the calculations of Eqs. (8) and (9) presented above.

3.3. Bayesian analysis for a random volatility IG process model

3.3.1. Model description

A random volatility IG process (RV) model is introduced by letting λ in the simple IG model follow a gamma distribution and the other parameters μ and $\Lambda(t)$ remain the same [21]. The RV model is then given as $Y_{RV}(t)\sim \mathrm{IG}(\mu\Lambda(t),\lambda\Lambda^2(t))$, $\lambda\sim\mathrm{Gamma}(\delta,\gamma)$, $\mu>0$, where the PDF of the gamma distribution is

$$g_\lambda(\lambda|\delta,\gamma)=\frac{\gamma^\delta\lambda^{\delta-1}}{\Gamma(\delta)}\exp(-\gamma\lambda),\delta>0,\gamma>0 \tag{19}$$

where $\Gamma(\delta)=\int_0^\infty t^{\delta-1}e^{-t}dt$ is the gamma function.

Similar to the RD model, the PDF of degradation process $Y_{RV}(t)$ with a RV model is obtained as

$$\begin{aligned}f_{RV}(y|\mu,\Lambda(t),\delta,\gamma)&=\int_{\lambda>0} f_S(y|\mu\Lambda(t),\lambda\Lambda^2(t))g_\lambda(\lambda|\delta,\gamma)d\lambda\\ &=\frac{\Gamma(1/2+\delta)\gamma^\delta}{\Gamma(\delta)\left[\gamma+(y-\mu\Lambda(t))^2/(2\mu^2y)\right]^{1/2+\delta}}\sqrt{\frac{\Lambda^2(t)}{2\pi y}}\end{aligned} \tag{20}$$

The reliability function with failure threshold D for the RV model is given as

$$R_{RV}(t|\mu,\Lambda(t),\delta,\gamma)=\int_{\lambda>0} R_S(y|\mu\Lambda(t),\lambda\Lambda^2(t))g_\lambda(\lambda|\delta,\gamma)d\lambda$$

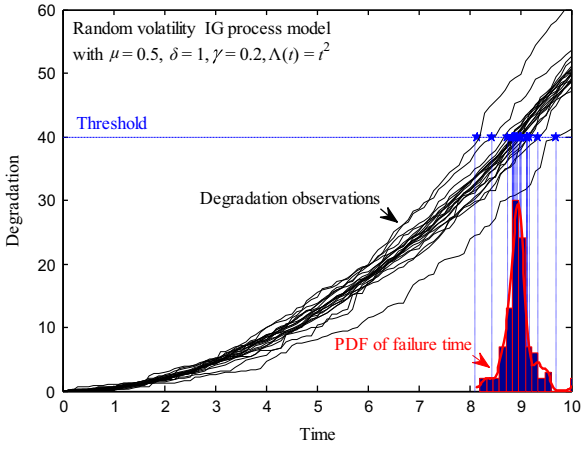

Fig. 6. A simulated RV model with threshold $D=40$.

$$=\int_{\lambda>0}\left\{\Phi\left[\sqrt{\frac{\lambda}{D}}\left(\frac{D}{\mu}-\Lambda(t)\right)\right]+\exp\left(\frac{2\lambda\Lambda(t)}{\mu}\right)\times\Phi\left[-\sqrt{\frac{\lambda}{D}}\left(\frac{D}{\mu}+\Lambda(t)\right)\right]\right\}\frac{\gamma^\delta\lambda^{\delta-1}}{\Gamma(\delta)}\exp(-\gamma\lambda)d\lambda \tag{21}$$

A group of simulated degradation paths of a RV model are presented in Fig. 6. As random effects are involved in the scale parameter λ, the variance of degradation observations within each unit is significant, yet the degradation rates among these units are relevantly coherent. It can be identified by comparing the smoothness of relevant degradation curves with the ones in Fig. 5. Accordingly, the RV model is suitable for degradation modeling of products for which overall degradation rate is coherent and obvious unit-specific degradation variation exists.

3.3.2. Bayesian analysis

Similar to Section 3.2.2, under the RV model, the degradation increments $\Delta y_{ij}=Y_{RV}(t_{ij})-Y_{RV}(t_{ij-1})$ are independent and follow IG distribution $\mathrm{IG}(\mu\Delta\Lambda_{ij},\lambda_i\Delta\Lambda_{ij}^2)$ with $\lambda_i\sim\mathrm{Gamma}(\delta,\gamma)$. Prior distributions for parameters μ, θ_Λ, δ, and γ are specified following the hierarchical strategy as

$$\begin{cases}\delta\sim\mathrm{Gamma}(a_\delta,b_\delta)\\ \gamma\sim\mathrm{Gamma}(a_\gamma,b_\gamma)\end{cases},\begin{cases}\mu\sim\mathrm{TN}(\omega,\kappa^{-2})\\ \theta_\Lambda\sim\mathrm{Uniform}(a_\Lambda,b_\Lambda)\end{cases} \tag{22}$$

Similarly, the likelihood function for the degradation data Y_{RV} under RV model is obtained as

$$L_{RV}(Y_{RV},\boldsymbol{\lambda}|\mu,\theta_\Lambda,\delta,\gamma)=\prod_{i=1}^{n}\left\{g_\lambda(\lambda_i|\delta,\gamma)\prod_{j=1}^{m_i}f(\Delta y_{ij}|\mu\Delta\Lambda_{ij},\lambda_i\Delta\Lambda_{ij}^2)\right\} \tag{23}$$

where $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)$ includes all the random parameters for the degradation paths Y_{RV}.

Based on the prior distributions and the likelihood function given above, the joint posterior distribution for both fixed parameters μ, θ_Λ, δ and γ, and random parameters $\boldsymbol{\lambda}$ is obtained as

$$\begin{aligned}p(\mu,\theta_\Lambda,\delta,\gamma,\boldsymbol{\lambda}|Y_{RV})&\propto\pi(\mu)\pi(\theta_\Lambda)\pi(\delta)\pi(\gamma)L_{RV}(Y_{RV},\boldsymbol{\lambda}|\mu,\theta_\Lambda,\delta,\gamma)\\ &\propto\phi[\kappa(\mu-\omega)]\delta^{a_\delta-1}\exp(-b_\delta\delta)\gamma^{a_\gamma-1}\exp(-b_\gamma\gamma)\times\\ &\prod_{i=1}^{n}\left\{\frac{\gamma^\delta\lambda^{\delta-1}}{\Gamma(\delta)}\exp(-\gamma\lambda)\prod_{j=1}^{m_i}\sqrt{\lambda\Delta\Lambda_{ij}^2}\exp\left[-\frac{\lambda(\Delta y_{ij}-\mu_i\Delta\Lambda_{ij})^2}{2\mu_i^2\Delta y_{ij}}\right]\right\}\end{aligned} \tag{24}$$

Based on the joint posterior distribution, the inference of reliability for the product population and the prediction of degradation

for the ith product are obtained as

$$R_{RV}(t|Y_{RV}) = \int_{\mu,\theta_\Lambda,\delta,\gamma} R_{RV}(t|\mu,\Lambda(t),\delta,\gamma)p(\mu,\theta_\Lambda,\delta,\gamma|Y_{RV})d\mu d\theta_\Lambda d\delta d\gamma \quad (25)$$

$$f_{RVi,m+1}(y|Y_{RV}) = \int_{\mu,\theta_\Lambda,\lambda_i} f_S(y|\mu\Delta\Lambda_{ij},\lambda_i\Delta\Lambda_{ij}^2)p(\mu,\theta_\Lambda,\lambda_i|Y_{RV})d\mu d\theta_\Lambda d\lambda_i \quad (26)$$

where $f_{RVi,m+1}(y|Y_{RV})$ is the predicted PDF of degradation for the ith product at time point $t_{i,m+1}$, and $p(\mu,\theta_\Lambda,\delta,\gamma|Y_{RV})$ and $p(\mu,\theta_\Lambda,\lambda_i|Y_{RV})$ are the marginal joint posterior distributions for the corresponding parameters, which are obtained from the joint posterior distribution given in Eq. (24) by integrating over the parameters being excluded.

Similar to Section 3.2.2, the MCMC method and the simulation based integration strategy are used for the calculations above. The OpenBUGS code script for this model is obtained by slightly modifying the one for the RD model provided in Appendix.

3.4. Bayesian analysis for a random drift-volatility IG process model

3.4.1. Model description

To account for the consideration that a unit with a higher degradation rate is expected to suffer higher degradation variances, Ye and Chen [21] introduce the random drift-volatility IG process (RDV) model. It is given as $Y_{RDV}(t) \sim \mathrm{IG}(\mu\Lambda(t),\lambda\mu^2\Lambda^2(t))$, $\mu \sim TN(\omega,\kappa^{-2})$, $\lambda > 0$. Similar to Section 3.2, the PDF of the RDV model is obtained as

$$f_{RDV}(y|\omega,\kappa,\Lambda(t),\lambda) = \int_{\mu>0}\sqrt{\frac{\lambda\mu^2\Lambda^2(t)}{2\pi y^3}}\exp\left[-\frac{\lambda(y-\mu\Lambda(t))^2}{2y}\right]\frac{\kappa\phi[\kappa(\mu-\omega)]}{1-\Phi(-\kappa\omega)}d\mu \quad (27)$$

The reliability function with failure threshold D for this model is also given as

$$R_{RDV}(t|\omega,\kappa,\Lambda(t),\lambda) = \int_{\mu>0}\left\{\Phi\left[\sqrt{\frac{\lambda}{D}}(D-\mu\Lambda(t))\right] + e^{2\lambda\mu\Lambda(t)}\Phi\left[-\sqrt{\frac{\lambda}{D}}(D+\mu\Lambda(t))\right]\right\}\frac{\kappa\phi[\kappa(\mu-\omega)]}{1-\Phi(-\kappa\omega)}d\mu \quad (28)$$

An intuitive description of the degradation paths simulated from a RDV model is presented in Fig. 7. Since the random effects are involved in the degradation mean and variance, these degradation paths vary significantly. Both the degradation rates among product population and the variance within specific unit are embodied with random effects. This model is suitable for degradation modeling of products where significant product heterogeneity is observed, such as the products subjected to different working conditions or operation frequencies (e.g., [22]).

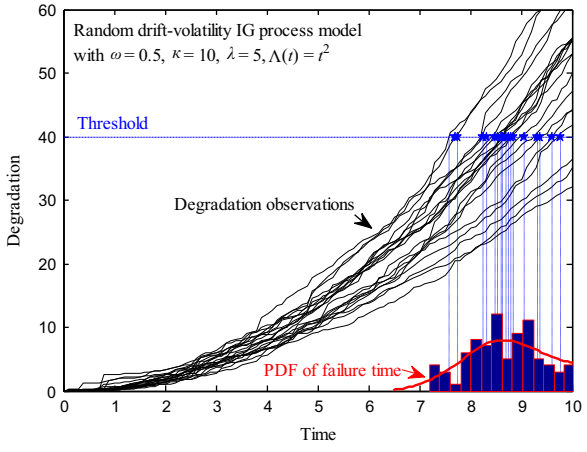

Fig. 7. A simulated RDV model with threshold $D = 40$.

3.4.2. Bayesian analysis

Since random effects are introduced through parameter μ, the Bayesian analysis of this RDV model is the same as the procedure for RD model described in Section 3.2.2. The only difference is that the likelihood function is given as

$$L_{RDV}(Y_{RDV},\boldsymbol{\mu}|\omega,\kappa,\theta_\Lambda,\lambda) = \prod_{i=1}^{n}\left\{g_\mu(\mu_i|\omega,\kappa^{-2})\prod_{j=1}^{m_i}f(\Delta y_{ij}|\mu_i\Delta\Lambda_{ij},\lambda\mu_i^2\Delta\Lambda_{ij}^2)\right\} \quad (29)$$

where $\boldsymbol{\mu} = (\mu_1,\ldots,\mu_n)$ includes all the random parameters for the degradation paths Y_{RDV}. The $g_\mu(\mu_i|\omega,\kappa^{-2})\prod_{j=1}^{m_i}f(\Delta y_{ij}|\mu_i\Delta\Lambda_{ij},\lambda\mu_i^2\Delta\Lambda_{ij}^2)$ is the likelihood contribution of the ith degradation path $Y_{RDV,i}$. It is presented in Fig. 2 with $f(\Delta y_{ij}|\mu_i\Delta\Lambda_{ij},\lambda\mu_i^2\Delta\Lambda_{ij}^2)$ given as $f(\Delta y_{ij}|\boldsymbol{\theta}^F,\boldsymbol{\theta}_i^R)$, and $g_\mu(\mu_i|\omega,\kappa^{-2})$ given as $\pi(\boldsymbol{\theta}_i^R|\boldsymbol{\theta}^H)$.

The joint posterior distribution and the corresponding posterior inference and prediction with this model are obtained as

$$\begin{aligned} p(\omega,\kappa,\theta_\Lambda,\lambda,\boldsymbol{\mu}|Y_{RDV}) &\propto \pi(\omega)\pi(\kappa)\pi(\theta_\Lambda)\pi(\lambda)L_{RDV}(Y_{RDV},\boldsymbol{\mu}|\omega,\kappa,\theta_\Lambda,\lambda) \\ &\propto \phi\left[b_\omega(\omega-a_\omega)\right]\phi\left[b_\kappa(\kappa-a_\kappa)\right]\lambda^{\delta-1}\exp(-\gamma\lambda)\times \\ &\prod_{i=1}^{n}\left\{\frac{\kappa\phi\left[\kappa(\mu_i-\omega)\right]}{1-\Phi(-\kappa\omega)}\prod_{j=1}^{m_i}\sqrt{\lambda\mu_i^2\Delta\Lambda_{ij}^2}\exp\left[-\frac{\lambda(\Delta y_{ij}-\mu_i\Delta\Lambda_{ij})^2}{2\Delta y_{ij}}\right]\right\} \end{aligned} \quad (30)$$

$$R_{RDV}(t|Y_{RDV}) = \int_{\omega,\kappa,\theta_\Lambda,\lambda} R_{RDV}(t|\omega,\kappa,\Lambda(t),\lambda)p(\omega,\kappa,\theta_\Lambda,\lambda|Y_{RDV})d\omega d\kappa d\theta_\Lambda d\lambda \quad (31)$$

$$f_{RDVi,m+1}(y|Y_{RDV}) = \int_{\mu_i,q,\lambda} f_S(y|\mu_i\Lambda(t),\lambda\mu_i^2\Lambda^2(t))p\left(\mu_i,\theta_\Lambda,\lambda|Y_{RDV}\right)d\mu_i d\theta_\Lambda d\lambda \quad (32)$$

where $f_{RDVi,m+1}(y|Y_{RDV})$ is the predicted PDF of degradation for the ith product at time point $t_{i,m+1}$.

Similarly, the MCMC method and a simulation based integration strategy are used to obtain the relevant indexes in Eqs. (31) and (32). The OpenBUGS code for this model can be constructed from the one for the RD model by modifying the code for parameters of the IG distribution.

4. Goodness-of-fit test

The inference of reliability and prediction of degradation are based on the generated posterior samples of model parameters. To present a reliable posterior analysis, it is necessary to test the goodness-of-fit of relevant IG process models within the Bayesian framework. Two simple graphical methods and two quantitative methods have been introduced to implement goodness-of-fit test for the IG process models by Wang and Xu [19] and Ye and Chen [21]. In this subsection, we describe a model diagnostic approach specified on the proposed Bayesian framework, which is based on the posterior samples generated through the MCMC. This method was delivered through the Bayesian χ^2 goodness-of-fit test proposed by Johnson [39]. A specific procedure for the RD model introduced by Ye and Chen [21] was presented in our previous study [27]. We then generalize this Bayesian χ^2 goodness-of-fit test for the IG process models by making the functions and procedures applicable for all the IG modes in this section.

For degradation data $Y_A = \{Y_{A,1},\ldots,Y_{A,n}\}$, the degradation increments Δy_{ij} are independent samples from relevant IG distributions with CDF $F_A(y|\theta_A)$, where A stands for the relevant IG process models described above and the parameters θ_A are the same for these degradation increments. Let $0 = a_0 < a_1 < \cdots < a_K = 1$ denote K equally-spaced quantiles from a uniform

distribution, and define $p_l = a_l - a_{l-1}$, $l = 1, \ldots, K$ and $K \approx M^{0.4}$. M is the total number of degradation increments in the degradation data Y_A. The procedure of the Bayesian χ^2 goodness-of-fit test is described as follows.

a) Picking a generated random sample $\tilde{\theta}_A$ from the joint posterior distribution $p_A(\theta_A|Y_A)$;
b) Calculating the number of degradation increments Δy_{ij}, $j = 1, \ldots, m_i$, $i = 1, \ldots, n$ that fall into the interval $[a_{l-1}, a_l]$ with $a_{l-1} < F_A(y_{ij}|\tilde{\theta}_A) \le a_l$ for all the K intervals as $m_l(\tilde{\theta}_A)$, $l = 1, \ldots, K$;
c) Calculating the Bayesian χ^2 test statistic for the random sample $\tilde{\theta}_A$ defined by

$$S(\tilde{\theta}_A) = \sum_{l=1}^{K} \frac{(m_l(\tilde{\theta}_A) - Mp_l)^2}{Mp_l} \tag{33}$$

d) Repeating steps (a) to (c) for L times and calculating the probability that $B_p = Pr(S(\tilde{\theta}_A) < \chi^2_{K-1,0.95})$ with $\chi^2_{K-1,0.95}$ denoting the 0.95 quantile of a chi-square distribution with $K-1$ degrees of freedom. L is generally chosen as the number of posterior samples generated by the MCMC.

The fitness of the model is calibrated by the probability B_p. It is intuitively reported as that B_p% of the generated posterior samples of parameters θ_A fit the data well. The higher this probability is, the better the model fits the data sets and the more effective the generated samples are used for posterior inference and prediction.

5. Simulation study

Let $(Y_{A,i}, T_i, m_i)$, $i = 1, \ldots, n$ be random degradation paths generated from an IG process model described above. $Y_{A,i}$, T_i, and m_i separately denote the degradation data, observation time points, and number of observation times for each degradation path. n is the sample size of the degradation model. We let sample size n separately equal to 5, 10, 15, 20, 30, 40, 50, 60, 80, and 100 to generate a group of degradation paths with gradually increasing populations. For each degradation path $(Y_{A,i}, T_i, m_i)$ in a population, the number of observation time points m_i is randomly chosen with $Pr(m_i = k) = 1/5$ and $k \in \{21, 22, 23, 24, 25\}$. Then the observation time points within T_i are created by randomly selecting m_i different points from $C_t = \{0.4, 0.8, 1.2, \ldots, 10\}$. Finally, the degradation data $Y_{A,i}$ are generated through sampling of degradation increases for specific IG process as $Y_A(t_{ij}) - Y_A(t_{i,j-1})$ with $t_{i,0} = 0$, $Y_A(t_{i,0}) = 0$, $j = 1, \ldots, m_i$, and $i = 1, \ldots, n$. In addition, to facilitate the presentation of the proposed Bayesian framework for the IG process models, we assume $\Lambda(t) = t^q$ with $q > 0$ as showed in Ye and Chen [21]. Various patterns of degradation process can be modeled through this power law function with different values of parameter q.

For the degradation data generated above, non-informative and informative prior distributions are separately applied in the Bayesian analysis. Following the principle of indifference, non-informative priors are given as uniform distribution with a large interval [30]. The length of this large interval is chosen as ten times of the true value of relevant parameters. For instance, the distributions $\omega \sim \text{Uniform}(0, 20)$, $\kappa \sim \text{Uniform}(0, 100)$, $q \sim \text{Uniform}(0, 12)$ and $\lambda \sim \text{Uniform}(0, 200)$ are chosen for the RDV model with parameters $\omega = 2$, $\kappa = 10$, $q = 1.2$, and $\lambda = 20$. Informative prior distributions are given in the form presented in Section 3 for each IG process model. Specifically, these prior distributions are obtained by letting the means of these distributions equal to true values and their variances equal to one quarter of true values of these parameters. For instance, $\omega \sim \text{TN}(2, 2^{-2})$, $\kappa \sim \text{TN}(10, 0.4^{-2})$, $q \sim \text{Uniform}(0.25, 2.15)$, and $\lambda \sim \text{Gamma}(16, 0.8)$ are chosen for the RDV IG process model with parameters $\omega = 2$, $\kappa = 10$, $q = 1.2$, and $\lambda = 20$. The prior distributions for other IG process models are given in the same way as the RDV model described above.

For the implementation of MCMC through OpenBUGS, it takes a certain number of iterations for samples generated from simulation runs are representative of a certain distribution. In our numerical analysis, we have used the practice of not using the first 5000 samples generated through simulation. Instead, the subsequent 10,000 samples are used. The convergence of every MCMC simulation is monitored by a build-in toolkit in OpenBUGS, which is based on the Gelman–Rubin ratio [40]. Based on these posterior samples generated through the MCMC simulation, statistical summaries of parameters are obtained. Considering the limitations of space, we only present the results of the RDV model in Table 1 and Fig. 8 for illustration.

Table 1 shows that parameters of the RDV model can be accurately estimated by the proposed Bayesian method under reasonable sample sizes. Fig. 8 describes the comparisons of estimation results between different sample sizes and priors. It shows that both the increase of sample size and the incorporation of informative priors can improve the precision of estimations.

In particular, for the parameters ω, q, and λ when sample size increases from five to fifty, the variances of estimations decrease a lot. Similarly, the informative priors work well for these parameters when sample size is small. However, for the parameter κ the incorporation of informative prior exerts more significant influence than the increase of sample size. In the scenario of non-informative priors, the estimation results of parameter κ reach a reasonable precision when sample size equals to forty. However, a sample size

Table 1
Estimated results of the RDV model with non-informative and informative priors. The true values of model parameters are $\omega = 2$, $\kappa = 10$, $q = 1.2$, and $\lambda = 20$.

Sample size	Param.	Non-informative priors				Informative priors			
		Posterior		Posterior probability intervals		Posterior		Posterior probability intervals	
		Mean	SD	2.5%	97.5%	Mean	SD	2.5%	97.5%
5	ω	2.15	0.136	1.88	2.433	2.011	0.027	1.762	2.3
	κ	14.1	6.109	4.676	29.35	10.45	2.263	6.23	14.99
	q	1.141	0.147	1.095	1.19	1.211	0.026	1.157	1.265
	λ	15.8	2.497	12.58	22.37	17.1	2.388	12.14	19.83
50	ω	2.039	0.043	1.96	2.121	2.023	0.009	1.937	2.112
	κ	8.811	4.192	6.462	11.94	9.635	1.629	7.29	12.51
	q	1.194	0.04	1.177	1.211	1.194	0.008	1.177	1.212
	λ	20.56	0.928	18.79	22.42	19.42	0.844	17.73	21.15
100	ω	2.003	0.029	1.936	2.071	2.011	0.006	1.954	2.068
	κ	10.16	1.124	7.996	12.95	11.58	0.909	9.301	14.3
	q	1.201	0.031	1.187	1.214	1.2	0.005	1.186	1.213
	λ	20.07	0.643	18.82	21.35	19.8	0.655	18.6	21.05

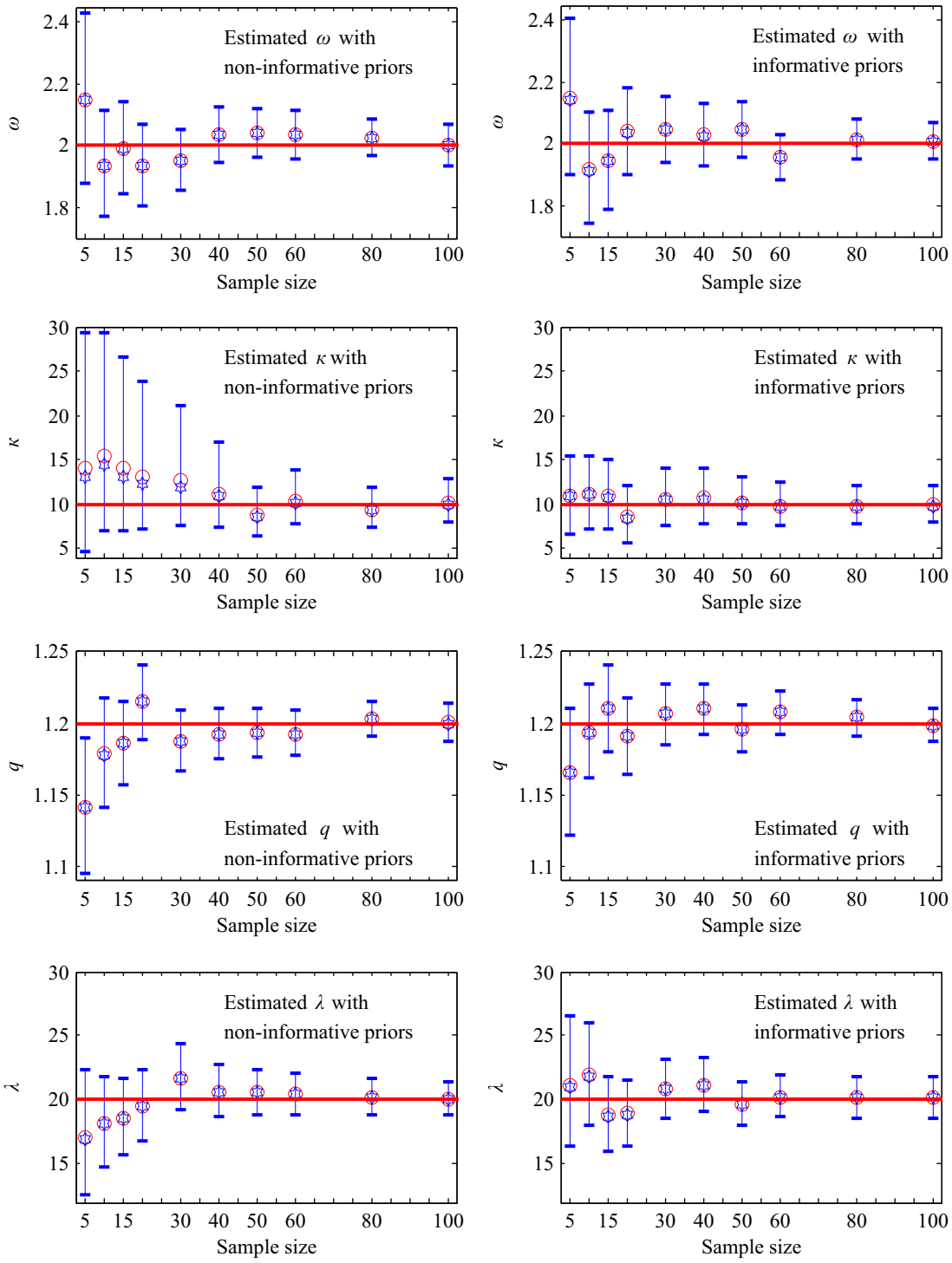

Fig. 8. Boxplot of estimation results of the RDV model.

of twenty can achieve this precision under the situation of informative priors. The advantage of incorporating informative priors can also be demonstrated on the rest of IG process models through simulation study. Accordingly, the incorporating of prior information can be an alternative of increasing sample size. The proposed Bayesian analysis of IG process models enjoys a natural way of incorporating subjective information through the framework described above.

As described in Section 3, the inference of reliability and prediction of degradation are obtained based on the samples generated by the MCMC. The corresponding inference of reliability is presented in Fig. 9. The prediction of the degradation $Y_{RDV}(t_{RDV,m+1}|Y)$ at time point $t_{RDV,m+1}=12$ is presented in Fig. 10.

6. An illustrative example

A GaAs laser degradation dataset is used to demonstrate the applicability of the proposed Bayesian method for degradation analysis with IG process models. This dataset has been investigated by Wang and Xu [19] and Ye and Chen [21] using the EM and bootstrap based MLE. It has been demonstrated by them that

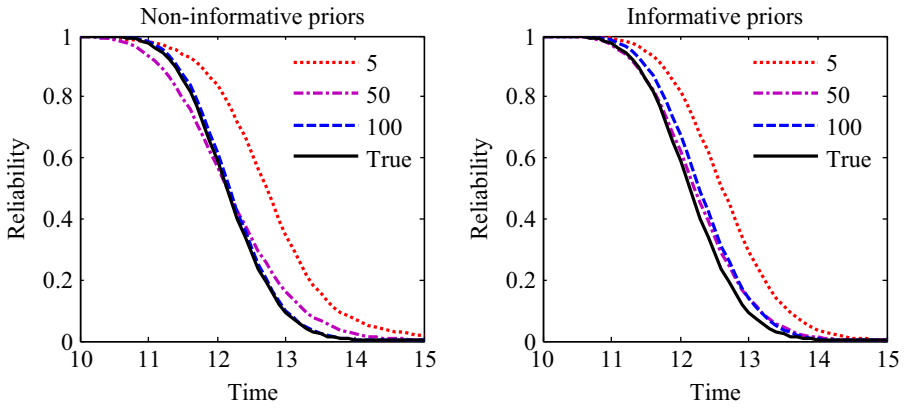

Fig. 9. Inferences of reliability for the RDV model with different sample sizes.

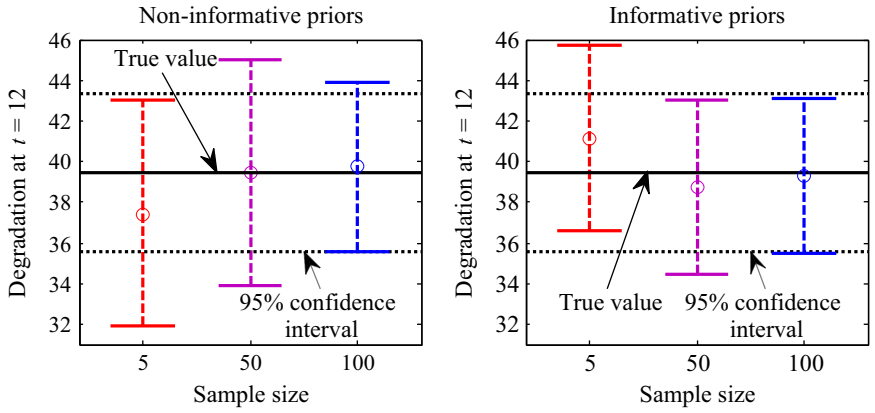

Fig. 10. Boxplot of predictions of degradation at $t_{RDV,m+1} = 12$ for the RDV model.

neither the Wiener nor the gamma process fits this dataset well. Moreover, the random drift IG process model introduced by Ye and Chen [21] provides a good fit among other IG process models introduced above. In this section, we further investigate this dataset with the family of IG process models using the proposed Bayesian method.

For a GaAs Laser device, light output will degrade as service time increases. Increasing operating current may complement this inherent degradation to maintain a constant light output. However, it will fail when this operating current crosses a predefined threshold which it cannot bear. The degradation dataset describes the increase of operating current over time for 15 GaAs laser devices [2]. In detail, this dataset is presented in the form as (Y_i, T_i, m_i), $i = 1, \ldots, n$. The sample size n is 15. The observation time points T_i are the same for all the samples with $m_i = 16$ and $T_i = \{0.25, 0.5, 0.75, \ldots, 4\}$ thousands hours for $i = 1, \ldots, 15$. The degradation increases Δy_{ij}, $i = 1, \ldots, 15$, $j = 1, \ldots, 16$ which measure the percentage increases in operating current are plotted in Fig. 11. For specific data please refer to Table C.17 in Meeker and Escobar [2].

The IG process models described in Section 3 are applied to this dataset using the proposed Bayesian method. From the semi-parametric fitting in Wang and Xu [19] and parametric fitting in Ye and Chen [21], the assumption $\Lambda(t) = t^q, q > 0$ is used as well in this illustrative example. To obtain estimations which mainly depend on the observed data, the non-informative prior distributions are adopted for the IG process models. Similar to the simulation study, these non-informative priors are given in the form of uniform distributions with relevant large intervals. These intervals are determined based on the interpretations of these parameters. For instance, parameter μ in the simple IG process

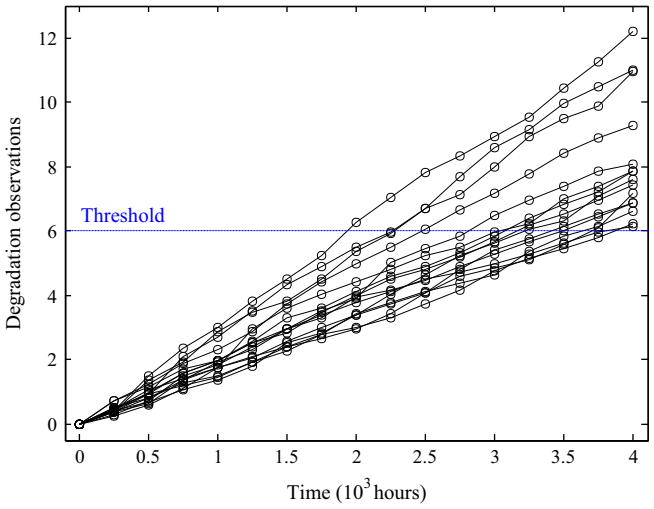

Fig. 11. Plot of the degradation dataset of GaAs laser devices with failure threshold as $D = 6\%$.

model is related to the degradation rate of a specific degradation curve. A uniform distribution with the interval $[0, 100]$ is diffuse enough to be chosen as a non-informative prior. It is because that few degradation rates could be larger than 100 per observation unit. Specific non-informative priors adopted in the case study are presented together with the estimation results in Table 2. The estimation results presented in Table 2 are summarized from the generated posterior samples. These samples are obtained

Table 2
Estimation results of the IG process models with non-informative priors. The posterior mean and 95% posterior probability intervals, and the intervals of uniform prior distributions are presented.

	Simple model					**RD model**			
	Mean	**2.5%**	**97.5%**	**Priors**		**Mean**	**2.5%**	**97.5%**	**Priors**
μ	2.172	1.929	2.440	[0, 100]	ω	2.189	1.882	2.512	[0, 100]
q	0.9672	0.9006	1.034	[0, 20]	κ	2.374	1.460	3.545	[0, 100]
λ						0.9649	0.9112	1.021	[0, 20]
					λ	0.1194	0.0931	0.1528	[0, 1000]
	RV model					**RDV model**			
	Mean	**2.5%**	**97.5%**	**Priors**		**Mean**	**2.5%**	**97.5%**	**Priors**
δ	675.8	106.6	987.7	[0, 1000]	ω	2.163	1.847	2.504	[0, 100]
γ	12.22	1.786	20.45	[0, 100]	κ	2.198	1.352	3.278	[0, 100]
μ	2.146	1.908	2.406	[0, 100]	q	0.9683	0.9131	1.025	[0, 20]
q	0.9746	0.9102	1.040	[0, 20]	λ	19.29	15.44	23.55	[0, 1000]

through the implementation of MCMC in OpenBUGS. For each model, the first 5000 samples are discarded and the subsequent 10,000 samples are used.

The estimation results are comparable with the results obtained using EM and bootstrap methods by Ye and Chen [21]. It indicates that the proposed Bayesian method is applicable for degradation data analysis with IG process models. In addition, estimation results of parameters δ and γ in RV model vary significantly, which is similar to the results by Wang and Xu [19] and Ye and Chen [21]. However, this significant diffuse of estimation results can be reduced by incorporating prior information about these parameter if available. Following the procedure given in Section 3.3, we have the following prior distributions $\delta \sim \text{Gamma}(4, 0.04)$ with mean 100 and standard deviance 50 and $\gamma \sim \text{Gamma}(4, 2)$ with mean 2 and standard deviance 1. By integrating these prior distributions with the observed degradation data set, the estimation confidence intervals of these parameters are obtained as [44.31, 193.2] for δ and [0.7542, 3.427] for γ. This improvement of estimation results is mainly due to the coherent integration of informative priors and degradation observations, where informative priors exert a significant effect on the final results. It is critical for the situation that the degradation observations are less informative for the parameters yet informative prior information is available. It is also an advantage of the proposed Bayesian method for degradation analysis with IG process models, which cannot be fulfilled with the EM and bootstrap methods.

In addition, boxplots of the posterior densities of parameters with random effects are presented in Fig. 12. A significant variation is observed for the parameter μ both in the RD model and in the RDV model. However, there is no obvious variation for the parameter λ in the RV model. The variation of parameter μ among product population is closely related to the variation of the degradation curves among the GaAs Laser population as presented in Fig. 11. This can be ascribed to the fact that the RD model and RDV model are suitable for the degradation process where significant variation of degradation rate is observed within a population. An intuitive selection of IG process models can be carried out based on the representing of variation among the GaAs Laser population.

The χ^2_1 quantile–quantile (Q–Q) plot and the Bayesian χ^2 goodness-of-fit test are implemented to select the best-fit IG process model for the dataset. The χ^2_1 Q–Q plots of the transformed degradation increments are presented in Fig. 13. It suggests that the RD model and the RDV model can present a relevant good fit to the dataset. In addition, the Bayesian χ^2 goodness-of-fit test is implemented to the four IG process models based on the generated posterior samples of relevant model parameters. The Bayesian χ^2 goodness-of-fit test statistics are separately 0.9813 for the simple IG process model, 1.00 for the RD model, 0.925 for the RV model, and 0.9947 for the RDV model. It indicates that nearly all the posterior samples of the RD process model fit the dataset well. Accordingly, the RD model is recommended both qualitative through the Q–Q plot and quantitatively from a Bayesian perspective. It is consistent with the conclusion obtained by Ye and Chen [21].

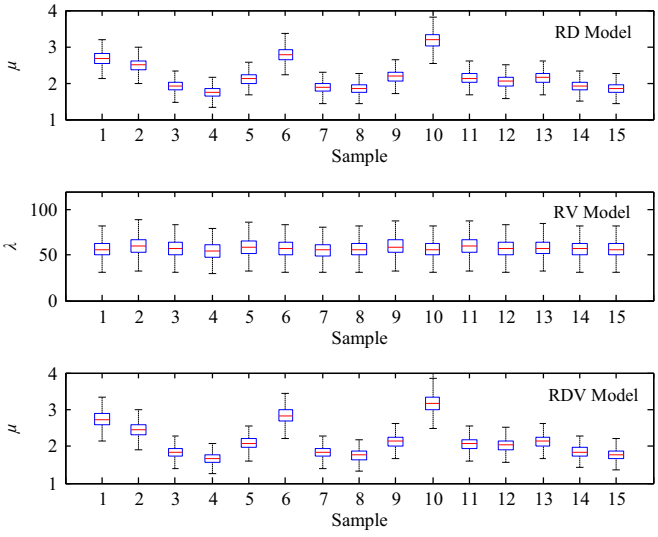

Fig. 12. Boxplot of the posterior densities of random parameters.

Since the RD model is chosen, the inference of reliability and the prediction of degradation are obtained based on its generated posterior samples through Eqs. (17) and (18). The inferences of reliability are presented in Fig. 14. To investigate the sensitivity of reliability to the threshold, four threshold levels are considered as $D = 6\%, 7\%, 8\%$, and 9%.

To demonstrate the flexibility of proposed Bayesian analysis for degradation prediction in the scenario of condition monitoring, a new degradation curve with limited observations is simulated and the prediction of degradation is obtained. As described in Fig. 3, the kernels of the posterior samples for model parameters of the RD model obtained above are incorporated as the prior distributions for the Bayesian analysis of this newly generated degradation data. The new degradation path is generated through Monte Carlo simulation using parameters estimated in Table 2. Four observations are generated at time points 250, 500, 750, and 1000 h. The prediction of degradation at time point 1250 h is obtained and presented in Table 3.

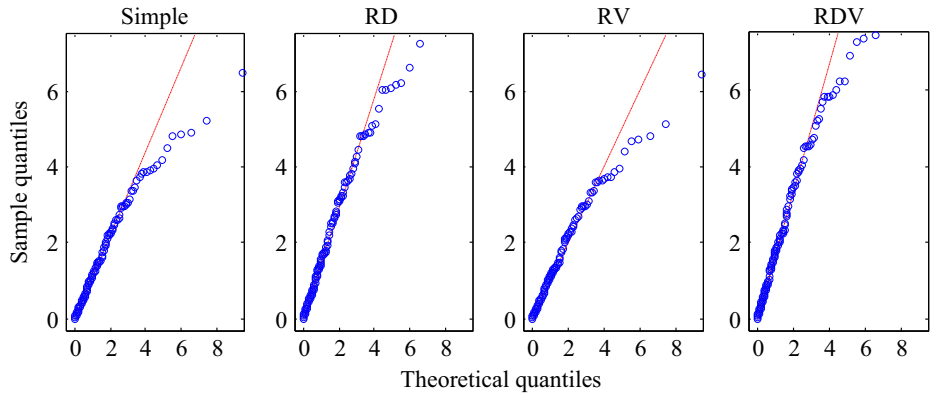

Fig. 13. Qualitative comparison between different IG process models using Q–Q plots of degradation increments of the GaAs laser dataset.

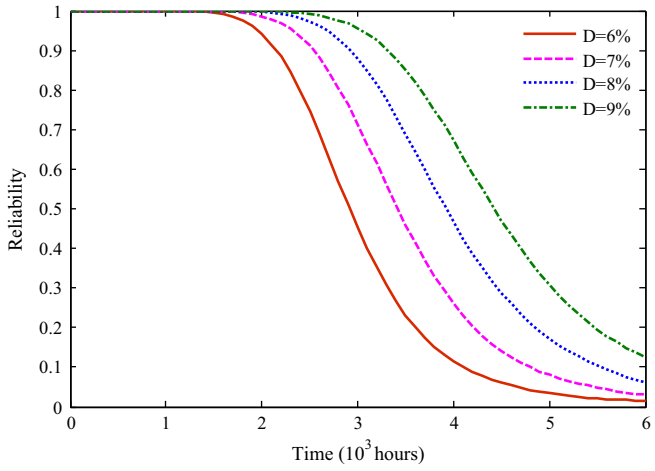

Fig. 14. Inferences of reliability with different threshold levels.

Table 3
Predicted degradation at time point 1250 h for newly generated degradation curve.

Newly generated degradation through MC					Predicted degradation			
Time (h)	**250**	**500**	**750**	**1000**	**Mean**	**SD**	**2.5%**	**97.5%**
Degradation	0.461	1.276	1.921	2.362	2.660	0.584	1.497	3.831

7. Conclusions

This paper has systematically investigated the Bayesian method for degradation analysis with IG process models. The Bayesian analysis of degradation data with IG process models has various interesting features. It serves as an indispensable alternative for MLE-based degradation analysis with IG process models. It enjoys a natural way of incorporating subjective information within the degradation analysis, which is critical for the situation that degradation observations are scarce. In addition, the hierarchical specification of prior distributions makes the pooling of random effects information implemented in a practical and coherent way. Meanwhile, a simulation study has indicated that the incorporation of subjective information and the pooling of random effects information among product population are alternatives of increasing sample size. The flexibility of the proposed Bayesian degradation analysis with IG process models for on-line data analysis is highlighted through a classical case study.

Nevertheless, there are some aspects worth further investigation. For example, the degradation analysis using the IG process models with measurement errors is of interest for further study. Moreover, this Bayesian analysis of degradation with IG process model can be future applied to the fields highlighted in the introduction section, which includes the field of reliability tests, reliability analysis and fault prognostics.

Acknowledgement

This research was partially supported by the National Natural Science Foundation of China under the contract number 11272082, and the Fundamental Research Funds for the Central Universities under the contract number YBXSZC20131067. The authors are grateful to the anonymous reviewers for several helpful suggestions and comments that have improved the quality of the paper to a great extent.

Appendix

Code script of random drift IG process model in OpenBUGS

```
model{
#====Model Construction=============================================================
    for(i in 1 : nSample){                          # Sample size 'nSample' of degradation data 'Y'
                # Specify truncated normal distribution for random parameter 'mu[i]' of each degradation path
        mu[i] ~ dnorm(mu.a, mu.tau)I(0, )
        for(j in 2 : nTime[i]){     # Number of observations 'nTime[i]' for each degradation path 'Y[i]'
                            # Calcualte degradation increment 'deltaY[i, j]' with degradaton data 'Y[i, j]'
            deltaY[i, j] <- Y[i, j] - Y[i, j - 1]
                                                # Specify model parameters of IG distribution for
                                            # the degradation increment with the assumption 'Lambda(t) = t^q'
            deltaLambda[i, j] <- pow(t[i, j], q) - pow(t[i, j-1], q)          # Observation time point 't[i, j]'
            a[i, j] <- mu[i] * deltaLambda[i, j]
            b[i, j] <- pow(mu[i], 3) / lambda * pow(deltaLambda[i, j], 2)
            deltaY[i, j] ~ dinv.gauss(a[i, j], b[i, j])     # Specify IG distributions for the degradation increment
        }
    }
#====Prior Specification============================================================
                                                # Non-informative prior distributions in the form of
                                                # uniform distributions with large intervals
    mu.a ~ dunif(0, 100); mu.b ~ dunif(0, 100); mu.tau <- pow(mu.b, 2); # Priors for hyper-parameters
    q ~ dunif(0, 20); lambda ~ dunif(0, 1000)                           # Priors for model parameters
}
```

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