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An efficient approach to reliability-based design optimization within the enhanced sequential optimization and reliability assessment framework †

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Abstract

Reliability based design optimization (RBDO) has been widely implemented in engineering practices for high safety and reliability. It is an important challenge to improve computational efficiency. Sequential optimization and reliability assessment (SORA) has made great efforts to improve computational efficiency by decoupling a RBDO problem into sequential deterministic optimization and reliability analysis as a single-loop method. In this paper, in order to further improve computational efficiency and extend the application of the current SORA method, an enhanced SORA (ESORA) is proposed by considering constant and varying variances of random design variables while keeping the sequential framework. Some mathematical examples and an engineering case are given to illustrate the proposed method and validate the efficiency.

Keywords: Reliability based design optimization; Sequential optimization and reliability assessment; Computational efficiency; Reliability analysis; Single-loop method

1. Introduction

Reliability based design optimization (RBDO) is an approach to achieving reliable decision when considering the randomness of design variables and parameters which maybe come from manufacture, environment and so on [1-6]. The typical mathematical formulation of RBDO is as follows:

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X},\boldsymbol{\mu}_\mathbf{P}) \\ &s.t.\ \Pr(G_i(\mathbf{d},\mathbf{X},\mathbf{P}) \le 0) \ge \Phi(\beta_t),\ \ i = 1 \sim h \\ &\quad \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U,\ \boldsymbol{\mu}_\mathbf{X}^L \le \boldsymbol{\mu}_\mathbf{X} \le \boldsymbol{\mu}_\mathbf{X}^U \end{aligned} \tag{1}$$

where $\mathbf{d}$ is a vector of deterministic design variables; $\boldsymbol{\mu}_\mathbf{X}$ indicates a vector of mean values of random variables $\mathbf{X} = \{X_1, X_2, \cdots, X_n\}$ while $\boldsymbol{\mu}_\mathbf{P}$ represents a vector of mean values of random parameters $\mathbf{P} = \{P_1, P_2, \cdots, P_m\}$. $f(\cdot)$ is the objective function. $G_i(\cdot)$, $i = 1 \sim h$ are performance functions, and $\Pr(G_i(\cdot) \le 0)$ is the probability of success. $\Phi(\beta_t)$ is the target reliability and β_t denotes the reliability index. $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal random variable. The superscripts 'L' and 'U' denote the lower and upper boundaries, respectively. In this formulation, $\mathbf{d},\boldsymbol{\mu}_\mathbf{X}$ are design variables.

Solving the RBDO problem directly will involve double loops: the outer loop and the inner loop. The outer loop is to minimize the objective function while reliability analysis is performed in the inner loop. To efficiently deal with RBDO problem, many methods are developed to improve the efficiency of reliability analysis, such as reliability index approach (RIA), performance measure approach (PMA) [7-9] and enhancing the efficiency of algorithm in finding the most probable point (MPP) [10-12]. Although the PMA can efficiently decrease computational expense in reliability analysis, the computational expense for the large-scale RBDO problem with double loop approach is still prohibited. Then two new classes for dealing with RBDO are proposed [13-18]. In the first class, the RBDO problem is decoupled into sequential deterministic optimization and reliability analysis [13-18]. When constructing the deterministic constraints in the deterministic optimization, the strategy of constraint shift is adopted in Ref. [13]; the sequential optimization and reliability assessment (SORA) proposed in Ref. [14] and enhanced one in Ref. [15] adopts the strategy that utilizing its MPP of previous cycle to obtain the shift vector of each random design variable to each probabilistic constraint. In the second class, the RBDO problem is converted into a deterministic optimization by eliminating the reliability analysis (performing in the inner loop) through the KKT condition [17, 18].

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The objective of this paper is to further improve the computational efficiency of the current SORA by considering both cases of constant and varying variances of random design variables while keeping the single-loop framework. The efficiency of the proposed method is compared with the existing approaches in Refs. [14, 16] with several illustrative examples.

This paper is organized as follows. In section 2, the SORA method is briefly reviewed as well as the PMA method. The enhanced SORA (ESORA) is proposed in section 3. Several examples are used to illustrate the efficiency of the proposed method in section 4, followed by the conclusions in section 5.

2. Review of SORA

2.1 Approach of PMA

Different from the deterministic optimization, the feasibility of probabilistic constraints needs to be checked when minimizing the objective function in RBDO. The *i*th probabilistic constraint is formulated as follows:

$$\Pr(G_i(\mathbf{d},\mathbf{X},\mathbf{P}) \le 0) = F_{G_i}(0) \ge \Phi(\beta_t) \tag{2}$$

where the cumulative distribution function $F_{G_i}(\cdot)$ is described as

$$F_{G_i}(0) = \int \cdots \int_{G_i(\cdot)\le 0} f_{\mathbf{X},\mathbf{P}}(\mathbf{x},\mathbf{p}) d\mathbf{x} d\mathbf{p} \tag{3}$$

where $f_{\mathbf{X},\mathbf{P}}(\mathbf{x},\mathbf{p})$ is the joint Probability Density Function (PDF) of $\mathbf{X},\mathbf{P}$ [9].

The first order reliability method (FORM) is a MPP-based method and has been widely used for reliability analysis of the RBDO. Based on the FORM, reliability index approach (RIA) and performance measure approach (PMA) are proposed [7, 20]. In the both approaches, the random variables and parameters $\mathbf{X},\mathbf{P}$ in X-space should be transformed into $\mathbf{U_X},\mathbf{U_P}$ in the standard normal space using the Rosenblatt transformation. It has been pointed out that the PMA is more stable than the RIA [7].

Based on the PMA, the probabilistic constraint in Eq. (2) can be expressed as $F_{G_i}^{-1}(\Phi(\beta_t)) \le 0$. So the RBDO formulation is rewritten as [9]:

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X},\boldsymbol{\mu}_\mathbf{P}) \\ &s.t.\ G_{p_i} = F_{G_i}^{-1}(\Phi(\beta_t)) \le 0,\ \ i = 1 \sim h \\ &\quad \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U,\ \boldsymbol{\mu}_\mathbf{X}^L \le \boldsymbol{\mu}_\mathbf{X} \le \boldsymbol{\mu}_\mathbf{X}^U \end{aligned} \tag{4}$$

where G_{p_i} can be obtained by

$$\begin{aligned} &G_{p_i} = \max_{\mathbf{U_X},\mathbf{U_P}} G(\mathbf{U_X},\mathbf{U_P}) \\ &s.t.\ \ \|\mathbf{U_X},\mathbf{U_P}\|_2 = \beta_t\ . \end{aligned} \tag{5}$$

The solutions of Eq. (5) are the MPP $\left(\mathbf{U}_\mathbf{X}^*,\mathbf{U}_\mathbf{P}^*\right)$ and the performance measure $G_p = G(\mathbf{U}_\mathbf{X}^*,\mathbf{U}_\mathbf{P}^*)$. The MPP $\left(\mathbf{U}_\mathbf{X}^*,\mathbf{U}_\mathbf{P}^*\right)$ in the X-space can be obtained using the inverse Rosenblatt transformation. When the random variables follow the normal distribution, the MPP in X-space can be obtained by

$$\begin{aligned} X^* &= \mu_X + \sigma_X \cdot U_X^* \\ P^* &= \mu_P + \sigma_P \cdot U_P^*\ . \end{aligned} \tag{6}$$

After transformation, $G_p = G(\mathbf{U}_\mathbf{X}^*,\mathbf{U}_\mathbf{P}^*) = G(\mathbf{X}^*,\mathbf{P}^*)$. Eq. (4) can be rewritten as:

$$\begin{aligned} &\min_{\mathbf{d},\boldsymbol{\mu}_\mathbf{X}} f(\mathbf{d},\boldsymbol{\mu}_\mathbf{X},\boldsymbol{\mu}_\mathbf{P}) \\ &s.t.\ G_i(\mathbf{d},\mathbf{X}^{*,(i)},\mathbf{P}^{*,(i)}) \le 0,\ \ i = 1 \sim h \\ &\quad \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U,\ \boldsymbol{\mu}_\mathbf{X}^L \le \boldsymbol{\mu}_\mathbf{X} \le \boldsymbol{\mu}_\mathbf{X}^U\ . \end{aligned} \tag{7}$$

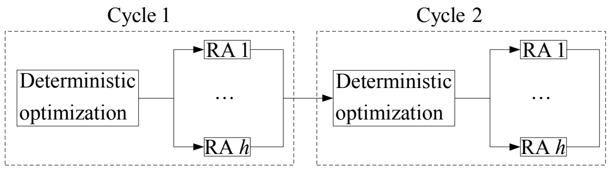

Fig. 1. Illustration of SORA (RA: reliability analysis).

2.2 Method of SORA

The SORA is one of the most efficient single-loop methods. The SORA decouples a RBDO problem into sequential deterministic optimization and reliability analysis as shown in Fig. 1 [14]. At each cycle, the deterministic optimization is first performed to obtain the optimum of each design variable, and then the reliability analysis of each probabilistic constraint is carried out at the optimal point. If all the probability constraints are not satisfied and the value of objective function is not stable, the MPPs information will be used in the next cycle. The flowchart of SORA is given in Fig. 2.

When the *i*th probabilistic constraint is violated in Cycle (*k*-1), it also means that the performance measure at the MPP does not satisfy $G_i(\mathbf{d},\mathbf{X}^{*,(i)},\mathbf{P}^{*,(i)}) \le 0$ with PMA. The SORA uses a strategy of "shift vector" to make sure the MPP of Cycle k falling into the deterministic feasible region $G_i(\mathbf{d},\mathbf{X}^{*,(i)},\mathbf{P}^{*,(i)}) \le 0$ [14]. The shift vector of the *i*th probabilistic constraint for Cycle *k* is:

$$\mathbf{s}^{(i),k} = \boldsymbol{\mu}_\mathbf{X}^{k-1} - \mathbf{X}^{*,(i),(k-1)} \tag{8}$$

where $\boldsymbol{\mu}_\mathbf{X}^{k-1}$ is the vector of mean values of random variables, and $\mathbf{X}^{*,(i),(k-1)}$ is the corresponding MPP of the *i*th probabilistic constraint obtained in cycle (*k*-1). It should be noted that each probabilistic constraint has its own shift vector because each one has its own MPP.

With the SORA method, the equivalent deterministic optimization of the original RBDO problem in cycle *k* is formulated as:

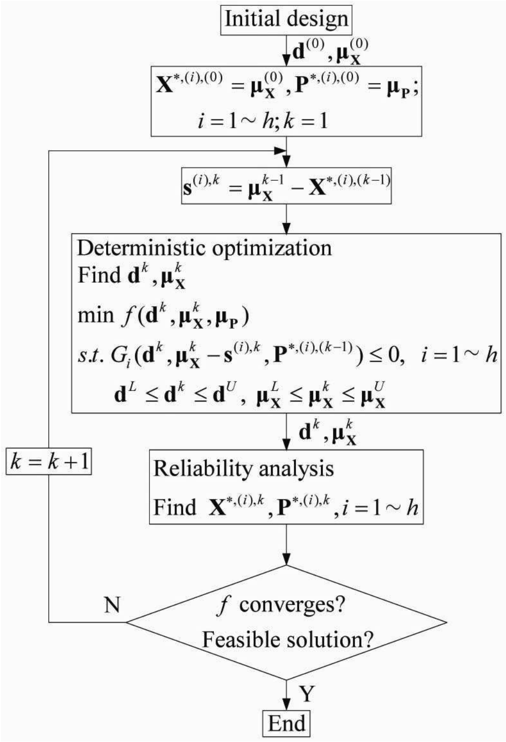

Fig. 2. Flowchart of SORA.

$$\begin{aligned}&\min_{\mathbf{d}^k,\boldsymbol{\mu}_\mathbf{X}^k} f(\mathbf{d}^k,\boldsymbol{\mu}_\mathbf{X}^k,\boldsymbol{\mu}_\mathbf{P})\\ &s.t.\ G_i(\mathbf{d}^k,\boldsymbol{\mu}_\mathbf{X}^k-\mathbf{s}^{(i),k},\mathbf{P}^{*,(i),(k-1)})\le 0,\ i=1\sim h\\ &\quad \mathbf{d}^L\le\mathbf{d}^k\le\mathbf{d}^U,\ \boldsymbol{\mu}_\mathbf{X}^L\le\boldsymbol{\mu}_\mathbf{X}^k\le\boldsymbol{\mu}_\mathbf{X}^U .\end{aligned}\tag{9}$$

From Eqs. (7)-(9), the MPP of Cycle k is approximated in the deterministic optimization in the SORA as:

$$\begin{aligned}X_j^{*,(i)}&\approx\mu_{X_j}^k-s_j^{(i),k}=\mu_{X_j}^k-(\mu_{X_j}^{k-1}-X_j^{*,(i),(k-1)})\\ &=\mu_{X_j}^k+\sigma_{X_j}\cdot U_{X_j}^{*,(i),(k-1)},j=1\sim n;\\ P_j^{*,(i)}&\approx P_j^{*,(i),(k-1)}\\ &=\mu_{P_j}+\sigma_{P_j}\cdot U_{P_j}^{*,(i),(k-1)},j=1\sim m .\end{aligned}\tag{10}$$

3. Enhanced SORA for RBDO problems

In this paper, it is assumed that each performance function is explicit so that its expression of gradient can be obtained. Based on Eq. (6), the following formulation holds in cycle k:

$$\begin{aligned}X_j^{*,(i),k}&=\mu_{X_j}^k+\sigma_{X_j}\cdot U_{X_j}^{*,(i),k},j=1\sim n\\ P_j^{*,(i),k}&=\mu_{P_j}+\sigma_{P_j}\cdot U_{P_j}^{*,(i),k},j=1\sim m .\end{aligned}\tag{11}$$

From Eqs. (10) and (11), the MPP is approximated in the U-space in the kth cycle of the SORA as:

$$\begin{aligned}\mathbf{U}_\mathbf{X}^{*,(i),k}&\approx\mathbf{U}_\mathbf{X}^{*,(i),(k-1)}\\ \mathbf{U}_\mathbf{P}^{*,(i),k}&\approx\mathbf{U}_\mathbf{P}^{*,(i),(k-1)} .\end{aligned}\tag{12}$$

With the PMA method, the MPP of a probabilistic constraint could be obtained:

$$\begin{aligned}&G_p=\max_{\mathbf{U}_\mathbf{X},\mathbf{U}_\mathbf{P}} G(\mathbf{U}_\mathbf{X},\mathbf{U}_\mathbf{P})\\ &s.t.\quad \|\mathbf{U}_\mathbf{X},\mathbf{U}_\mathbf{P}\|_2=\beta_t .\end{aligned}\tag{13}$$

Following the same way used in Ref. [17], based on Eq. (13) the relationship between the MPP $(\mathbf{U}_\mathbf{X}^*,\mathbf{U}_\mathbf{P}^*)$ and the gradient $\nabla G(\mathbf{U}_\mathbf{X},\mathbf{U}_\mathbf{P})|_{\mathbf{U}_\mathbf{X}^*,\mathbf{U}_\mathbf{P}^*}$ in the U-space is:

$$\begin{aligned}U_{X_j}^*&=\frac{\left.\partial G_\mathbf{U}(\mathbf{U}_\mathbf{X},\mathbf{U}_\mathbf{P})\big/\partial U_{X_j}\right|_{\mathbf{U}_\mathbf{X}^*,\mathbf{U}_\mathbf{P}^*}}{\|\nabla G_\mathbf{U}(\mathbf{U}_\mathbf{X}^*,\mathbf{U}_\mathbf{P}^*)\|_2}\cdot\beta_t,j=1\sim n\\ U_{P_j}^*&=\frac{\left.\partial G_\mathbf{U}(\mathbf{U}_\mathbf{X},\mathbf{U}_\mathbf{P})\big/\partial U_{P_j}\right|_{\mathbf{U}_\mathbf{X}^*,\mathbf{U}_\mathbf{P}^*}}{\|\nabla G_\mathbf{U}(\mathbf{U}_\mathbf{X}^*,\mathbf{U}_\mathbf{P}^*)\|_2}\cdot\beta_t,j=1\sim m\end{aligned}\tag{14}$$

and the relationship of gradient in the U-space and X-space is:

$$\begin{aligned}\frac{\partial G_\mathbf{U}(\mathbf{U}_\mathbf{X},\mathbf{U}_\mathbf{P})}{\partial U_{X_j}}&=\frac{\partial G_{\mathbf{X},\mathbf{P}}(\mathbf{d},\mathbf{X},\mathbf{P})}{\partial X_j}\cdot\sigma_{X_j},j=1\sim n\\ \frac{\partial G_\mathbf{U}(\mathbf{U}_\mathbf{X},\mathbf{U}_\mathbf{P})}{\partial U_{P_j}}&=\frac{\partial G_{\mathbf{X},\mathbf{P}}(\mathbf{d},\mathbf{X},\mathbf{P})}{\partial P_j}\cdot\sigma_{P_j},j=1\sim m .\end{aligned}\tag{15}$$

From Eqs. (6), (14) and (15), at the MPP $(\mathbf{X}^*,\mathbf{P}^*)$ in the X-space, the following formulation holds:

$$\begin{aligned}X_j^*&=\mu_{X_j}+\sigma_{X_j}\cdot\frac{b_{X_j}}{\|\mathbf{b}\|_2}\cdot\beta_t,j=1\sim n\\ P_j^*&=\mu_{P_j}+\sigma_{P_j}\cdot\frac{b_{P_j}}{\|\mathbf{b}\|_2}\cdot\beta_t,j=1\sim m\end{aligned}\tag{16}$$

where

$$\begin{aligned}&b_{X_j}=\left.\frac{\partial G(\mathbf{d},\mathbf{X},\mathbf{P})}{\partial X_j}\right|_{\mathbf{X}^*,\mathbf{P}^*}\cdot\sigma_{X_j},\mathbf{b}_\mathbf{X}=[b_{X_1},b_{X_2},\cdots,b_{X_n}]\\ &b_{P_j}=\left.\frac{\partial G(\mathbf{d},\mathbf{X},\mathbf{P})}{\partial P_j}\right|_{\mathbf{X}^*,\mathbf{P}^*}\cdot\sigma_{P_j},\mathbf{b}_\mathbf{P}=[b_{P_1},b_{P_2},\cdots,b_{P_m}]\\ &\mathbf{b}=[\mathbf{b}_\mathbf{X},\mathbf{b}_\mathbf{P}] .\end{aligned}$$

Eq. (16) holds for each cycle, and the MPP in Cycle k can be obtained by

$$\begin{aligned}X_j^{*,k}&=\mu_{X_j}^k+\sigma_{X_j}\cdot\frac{b_{X_j}^k}{\|\mathbf{b}^k\|_2}\cdot\beta_t,j=1\sim n\\ P_j^{*,k}&=\mu_{P_j}+\sigma_{P_j}\cdot\frac{b_{P_j}^k}{\|\mathbf{b}^k\|_2}\cdot\beta_t,j=1\sim m\end{aligned}\tag{17}$$

where

$$b_{X_j}^k = \left.\frac{\partial G(\mathbf{d},\mathbf{X},\mathbf{P})}{\partial X_j}\right|_{\mathbf{X}^{*,k},\mathbf{P}^{*,k}} \cdot \sigma_{X_j}, \mathbf{b}_{\mathbf{X}}^k = [b_{X_1}^k, b_{X_2}^k, \cdots, b_{X_n}^k]$$
$$b_{P_j}^k = \left.\frac{\partial G(\mathbf{d},\mathbf{X},\mathbf{P})}{\partial P_j}\right|_{\mathbf{X}^{*,k},\mathbf{P}^{*,k}} \cdot \sigma_{P_j}, \mathbf{b}_{\mathbf{P}}^k = [b_{P_1}^k, b_{P_2}^k, \cdots, b_{P_m}^k] \quad (18)$$
$$\mathbf{b}^k = [\mathbf{b}_{\mathbf{X}}^k, \mathbf{b}_{\mathbf{P}}^k].$$

When constructing the deterministic constraints in Cycle k, the gradient at the actual MPP can not be obtained because the reliability analysis is not performed. The gradient at the actual MPP is approximated in the following way.

For the random design variables with constant variances, from Eq. (12)

$$\frac{\mathbf{X}^{*,k} - \boldsymbol{\mu}_{\mathbf{X}}^k}{\boldsymbol{\sigma}_{\mathbf{X}}} \approx \frac{\mathbf{X}^{*,(k-1)} - \boldsymbol{\mu}_{\mathbf{X}}^{(k-1)}}{\boldsymbol{\sigma}_{\mathbf{X}}}$$

which is equivalent to

$$\mathbf{X}^{*,k} \approx \boldsymbol{\mu}_{\mathbf{X}}^k - \boldsymbol{\mu}_{\mathbf{X}}^{(k-1)} + \mathbf{X}^{*,(k-1)} \quad (19)$$

and the MPP of random parameters is:

$$\mathbf{P}^{*,k} \approx \mathbf{P}^{*,(k-1)}. \quad (20)$$

By substituting Eqs. (19) and (20) into Eq. (18), the approximation of the gradient at the MPP is obtained. At the first cycle $(k=1)$, the gradient at the actual MPP is approximated as the gradient at the mean values of random variables and parameters.

For the random design variables with varying variances, with Eqs. (14)-(18) the following formulation can be obtained:

$$X_j^{*,k} = \mu_{X_j}^k + \sigma_{X_j}^k \cdot \frac{b_{X_j}^k}{\|\mathbf{b}^k\|_2} \cdot \beta_t, j = 1 \sim n$$
$$P_j^{*,k} = \mu_{P_j} + \sigma_{P_j} \cdot \frac{b_{P_j}^k}{\|\mathbf{b}^k\|_2} \cdot \beta_t, j = 1 \sim m \quad (21)$$

where

$$b_{X_j}^k = \left.\frac{\partial G(\mathbf{d},\mathbf{X},\mathbf{P})}{\partial X_j}\right|_{\mathbf{X}^{*,k},\mathbf{P}^{*,k}} \cdot \sigma_{X_j}^k, \mathbf{b}_{\mathbf{X}}^k = [b_{X_1}^k, b_{X_2}^k, \cdots, b_{X_n}^k]$$
$$b_{P_j}^k = \left.\frac{\partial G(\mathbf{d},\mathbf{X},\mathbf{P})}{\partial P_j}\right|_{\mathbf{X}^{*,k},\mathbf{P}^{*,k}} \cdot \sigma_{P_j}, \mathbf{b}_{\mathbf{P}}^k = [b_{P_1}^k, b_{P_2}^k, \cdots, b_{P_m}^k] \quad (22)$$
$$\mathbf{b}^k = [\mathbf{b}_{\mathbf{X}}^k, \mathbf{b}_{\mathbf{P}}^k].$$

For the random design variables with varying variances (e.g. $\sigma = r \cdot \mu$, where r is the constant coefficient of variation), from Eq. (12)

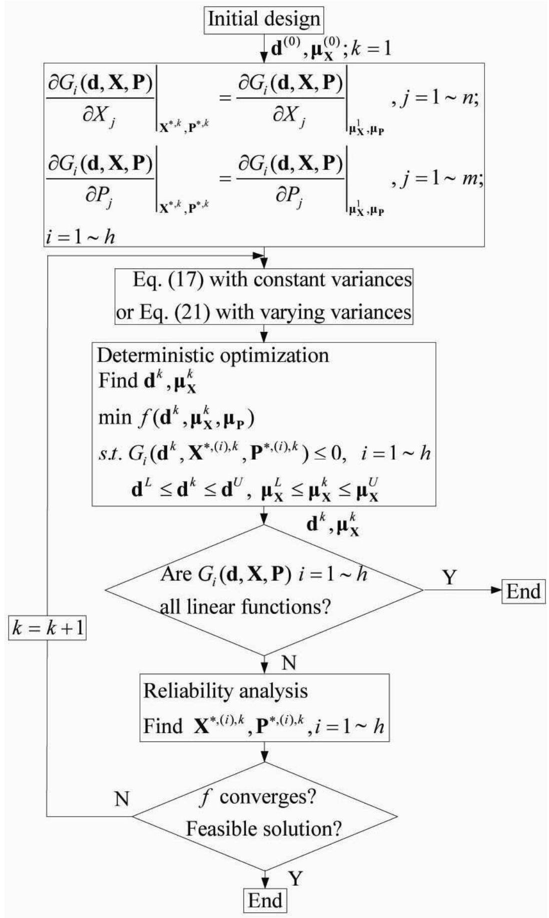

Fig. 3. Flowchart of the ESORA.

$$\frac{\mathbf{X}^{*,k} - \boldsymbol{\mu}_{\mathbf{X}}^k}{\boldsymbol{\sigma}_{\mathbf{X}}^k} \approx \frac{\mathbf{X}^{*,(k-1)} - \boldsymbol{\mu}_{\mathbf{X}}^{(k-1)}}{\boldsymbol{\sigma}_{\mathbf{X}}^{(k-1)}}$$

which is equivalent to

$$\mathbf{X}^{*,k} \approx \boldsymbol{\mu}_{\mathbf{X}}^k + \frac{\boldsymbol{\mu}_{\mathbf{X}}^k}{\boldsymbol{\mu}_{\mathbf{X}}^{(k-1)}} \cdot (\mathbf{X}^{*,(k-1)} - \boldsymbol{\mu}_{\mathbf{X}}^{(k-1)}). \quad (23)$$

By incorporating Eqs. (20) and (23) into Eq. (22), the gradient at the MPP is obtained. At the first cycle, the gradient at the actual MPP is approximated as the gradient at the mean values of random variables and parameters.

The discussion abovementioned is based on the normal random variables and parameters. For non-normal random variables and parameters, the Rackwitz-Fiessler' two-parameter equivalent normal method can be used to obtain the mean value and variance of equivalent normal distribution at a point of interest [16, 17]. Methods of choosing parameters for some non-normal random distributions to achieve the linear relationship between the mean value and standard deviation are proposed in Ref. [16].

The flowchart of the ESORA proposed in this paper is provided in Fig. 3. When the performance functions are all linear functions, because the gradient of each performance function is constant, the original RBDO problem is completely transformed into a deterministic optimization problem with Eqs. (17) and (21). In other words, the optimum of this deterministic optimization problem is the optimal solution of the original RBDO problem.

4. Numerical examples

In this section, two mathematical problems from Ref. [16] and a speed reducer design example from Ref. [20] are used to illustrate the efficiency of the proposed method. The optimal results of the proposed method are compared with those of the original SORA and the approaches in Ref. [16]. In this paper, all optimization problems are solved by means of the "fmincon" function in the software Matlab.

4.1 Problem 1 with linear constraints

In this problem, the objective function is nonlinear and the performance functions are linear. All input variables are normal distributed $(X_1 \sim X_6)$ with the distributions as $N(\mu_i, \sigma_i)$, $i = 1 \sim 6$. The target reliability is $0.99865 = \Phi(3)$ for each constraint [16]. The mathematical formulation of this RBDO is:

$$
\begin{aligned}
&\min f(\boldsymbol{\mu}_{\mathbf{X}}) = \frac{\mu_1\mu_2 - \mu_4^2}{\mu_3} - \sqrt{\mu_5\mu_6^3} \\
&s.t.\ P(g_i(x) \le 0) \ge R_i, \quad i = 1 \sim 4 \\
&g_1(x) = -(-X_1 + 3X_2 - 5) \\
&g_2(x) = -(-X_1 - 2X_3 - X_6 + 10) \\
&g_3(x) = -(X_1 + 2X_4 - X_5 - 8) \\
&g_4(x) = -(X_2 - 7X_6 + 2) \\
&1 \le \mu_1 \le 10, 2 \le \mu_2 \le 8, 3 \le \mu_3 \le 8 \\
&3 \le \mu_4 \le 8, 1 \le \mu_5 \le 6, 0.1 \le \mu_6 \le 2 .
\end{aligned} \tag{24}
$$

Optimums of the original SORA (Orig. SORA), approaches in Ref. [16], and the ESORA with varying variances are given in Table 1. The number of function evaluation (NFE) is also listed. In both cases of $r = 0.02$ and $r = 0.15$, the proposed ESORA efficiently solves the RBDO problem with the starting point as $[5\ 5\ 5\ 5\ 3\ 1]$, and the optimal solutions of ESORA is almost the same as those of Orig. SORA and approaches in Ref. [16]. Although the starting points and optimization method adopted in Ref. [16] are unknown, from Table 1 the NFE in the ESORA is obviously less than those of Orig. SORA and approaches in Ref. [16]. The reason is that in ESORA the original RBDO problem is completely transformed into a deterministic optimization because performance functions are all linear.

The results of two cases of constant variances are shown in Table 2. Because each performance function is linear, the NFE in the ESORA is obviously less than those needed in Orig. SORA and approaches in Ref. [16].

Table 1. Results and comparison for example 1 with varying variances.

	Approaches	x	f	Cycles	NFE
$r = 0.02$	Orig. SORA[a]	[1.0000, 8.0000, 3.0000, 8.0000, 6.0000, 1.3236]	-24.3472	4	185
	Approach 1[a]			3	149
	Approach 2[a]			3	149
	Approach 3[a]			3	149
	Double loop[a]			N/A	1804
	ESORA[b]	[1.0000, 8.0000, 3.0000, 8.0000, 6.0000, 1.3236]	-24.3472	N/A	71
$r = 0.15$	Orig. SORA[a]	[1.0000, 3.6479, 3.0000, 8.0000, 1.7444, 0.2603]	-20.1406	6	388
	Approach 1[a]			4	224
	Approach 2[a]			3	192
	Approach 3[a]			4	224
	Double loop[a]			N/A	1629
	ESORA[b]	[1.0000, 3.6488, 3.0000, 8.0000, 1.7434, 0.2603]	-20.1403	N/A	39

[a]Results from Ref. [16], [b]Results from our computation.

Table 2. Results and comparison for example 1 with constant variances.

	$\sigma_i = 0.02$		$\sigma_i = 0.15$	
	Cycles	NFE	Cycles	NFE
Orig. SORA[a]	3	135	3	135
Approach 1[a]				
Approach 2[a]				
Approach 3[a]				
Double loop[a]	N/A	1569	N/A	1793
ESORA[b]	N/A	63	N/A	47

[a]Results from Ref. [16], [b]Results from our computation.

4.2 Problem 2 with nonlinear constraints

In this problem, there are two normal random design variables X_1 and X_2 with the mean values of μ_1, μ_2 and the standard deviations of σ_1, σ_2. Three performance functions are all nonlinear and the target reliability is $0.99865 = \Phi(3)$ [16]. The mathematical formulation of this RBDO problem is:

$$
\begin{aligned}
&\min f(\boldsymbol{\mu}_{\mathbf{X}}) = \mu_1 + \mu_2 \\
&s.t.\ P(g_i(x) \le 0) \ge R_i \\
&g_1(x) = -(X_1^2 + \frac{X_2}{20} - 1) \\
&g_2(x) = -\left[\frac{(X_1 + X_2 - 5)^2}{30} + \frac{(X_1 - X_2 - 12)^2}{120} - 1\right] \\
&g_3(x) = -(\frac{80}{X_1^2 + 8X_2 + 5} - 1) \\
&0.01 \le \mu_i \le 10, \quad i = 1,2 .
\end{aligned} \tag{25}
$$

Table 3. Optimal results of example 2 with constant variances.

Design variables	$\sigma_i = 0.02$	$\sigma_i = 0.20$	$\sigma_i = 0.40$
	ESORA	ESORA	ESORA
μ_1	1.0598	1.5999	2.1282
μ_2	0.0100	0.0100	2.7788
Objective	1.0698	1.6099	4.9070
Constraints			
g_1	-0.2544	-1.5302	-3.6083
g_2	-0.4872	-0.0611	-5.8340×10^{-11}
g_3	-10.9405	-5.2181	-0.8593
Cycles	2	2	2
NFE	144	233	265

Table 4. Optimal results of example 2 with varying variances.

Design variables	$r_i = 0.02$	$r_i = 0.10$	$r_i = 0.20$
	ESORA	ESORA	ESORA
μ_1	1.0636	1.4282	2.2461
μ_2	0.0100	0.0100	3.8922
Objective	1.0736	1.4382	6.1383
Constraints			
g_1	-0.2715	-2.4478	-4.1228
g_2	-0.4843	-0.1863	-2.0610×10-8
g_3	-11.5963	-8.3815	-0.3109
Cycles	2	2	4
NFE	96	92	350

[a]Results from Ref. [16], [b]Results from our computation.

With the numerical tolerance of 0.01%, the optimal solutions of ESORA with the cases of constant and varying variances are provided in Tables 3 and 4 respectively. From Tables 3 and 4, when the constant variances and the constant coefficient of variation increase, the NFE tends to increase and also the objective value. The performance measure at MPP of each probabilistic constraint is less than zero, which indicates that each design point is feasible.

Tables 5 and 6 provide the cycles and NFE needed in the Orig. SORA, approaches in Ref. [16], and the ESORA. The ESORA efficiently solves the RBDO problem in two cycles for different cases of constant variances. The ESORA is more efficient than the double loop method since the NFE of the ESORA is less than that of double loop method. The cycles needed in the ESORA are all no more than those of Orig. SORA, and approaches in Ref. [16].

4.3 Speed reducer design example

This example is derived from Ref [20], shown in Fig. 4. In this paper, it is modified as a RBDO problem including six random design variables and one deterministic design variable. The properties of design variables of the speed reducer are given in Fig. 7. All random variables are normally distributed.

Table 5. Comparisons of different approaches with constant variances for example 2.

	$\sigma_i = 0.02$		$\sigma_i = 0.20$		$\sigma_i = 0.40$	
	Cycles	NFE	Cycles	NFE	Cycles	NFE
Orig. SORA[a]	3	87	3	117	4	295
Approach 1[a]						
Approach 2[a]						
Approach 3[a]						
Double loop[a]	N/A	491	N/A	644	N/A	1004
ESORA[b]	2	144	2	233	2	265

Table 6. Comparison of different approaches with varying variances for example 2.

	$r_i = 0.02$		$r_i = 0.10$		$r_i = 0.20$	
	Cycles	NFE	Cycles	NFE	Cycles	NFE
Orig. SORA[a]	6	141	13	289	18	795
Approach 1[a]	3	81	3	90	6	369
Approach 2[a]	3	81	3	90	4	268
Approach 3[a]	3	81	3	90	5	329
Double loop[a]	N/A	491	N/A	626	N/A	794
ESORA[b]	2	96	2	92	4	350

[a]Results from Ref. [16], [b]Results from our computation.

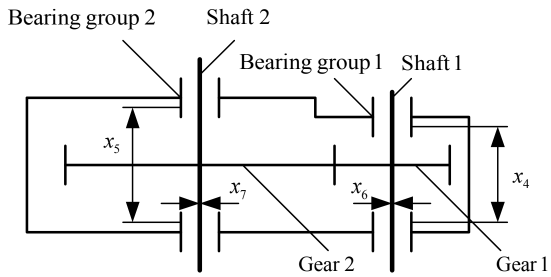

Fig. 4. Design variables of the speed reducer design.

The system objective function f is the speed reducer volume to be minimized. The RBDO formulation is:

$$\min \quad f = 0.7854x_1x_2^2\left(3.333x_3^2 + 14.933x_3 - 43.0934\right) - 1.508x_1\left(\mu_{x_6}^2 + \mu_{x_7}^2\right) + 7.477\left(\mu_{x_6}^3 + \mu_{x_7}^3\right) + 0.7854\left(\mu_{x_4}\mu_{x_6}^2 + \mu_{x_5}\mu_{x_7}^2\right)$$

$$\text{s.t.} \quad P\left(g_i \le 0\right) \ge R_i, i = 1 \sim 11$$

where

$g_1 = 27/\left(x_1x_2^2x_3\right) - 1 \le 0$: Upper bound on the bending stress of the gear tooth.

$g_2 = 397.5/(x_1x_2^2x_3^2) - 1 \le 0$: Upper bound on the contact stress of the gear tooth.

$g_3 = 1.93\mu_{x_4}^3/(x_2x_3\mu_{x_6}^4) - 1 \le 0$: Upper bound on the transverse deflection of the shaft 1.

$g_4 = 1.93\mu_{x_5}^3/(x_2x_3\mu_{x_7}^4) - 1 \le 0$: Upper bound on the trans-

Table 7. Distribution details of random design variables in the speed reducer design.

Deterministic & random variables	Description	Standard deviation	Distribution	Lower bound	Upper bound
μ_{x_1}	Mean of gear face width	0.01	Normal	2.6	3.6
μ_{x_2}	Mean of teeth module	0.01	Normal	0.3	1.0
x_3	Number of teeth of pinion	-	-	17	28
μ_{x_4}	Mean of distance between bearings 1	0.02	Normal	7.3	8.3
μ_{x_5}	Mean of distance between bearings 2	0.03	Normal	7.3	8.3
μ_{x_6}	Mean of diameter of shaft 1	0.005	Normal	2.9	3.9
μ_{x_7}	Mean of diameter of shaft 2	0.005	Normal	5	5.5

verse deflection of the shaft 2:

$g_5 = A_1 / B_1 - 1100 \le 0$: Upper bound on the stresses of the shaft 1:

$g_6 = A_2 / B_2 - 850 \le 0$: Upper bound on the stresses of the shaft 2:

$g_7 = x_2 x_3 - 40 \le 0$, $g_8 = x_1 / x_2 - 12 \le 0$ and $g_9 = -x_1 / x_2 + 5 \le 0$: Dimensional restrictions based on space;

$g_{10} = (1.5\mu_{x_6} + 1.9)/\mu_{x_4} - 1 \le 0$: Design condition for the shaft 1 based on experiences:

$g_{11} = (1.1\mu_{x_7} + 1.9)/\mu_{x_5} - 1 \le 0$: Design condition for the shaft 2 based on experiences.

where $A_1 = \left[\left(\dfrac{745\mu_{x_4}}{x_2 x_3}\right)^2 + 16.9\times 10^6\right]^{0.5}$, $B_1 = 0.1\mu_{x_6}^3$ and $A_2 = \left[\left(\dfrac{745\mu_{x_5}}{x_2 x_3}\right)^2 + 157.5\times 10^6\right]^{0.5}$, $B_2 = 0.1\mu_{x_7}^3$.

The target reliability is $0.99865 = \Phi(3)$ for each probabilistic constraint. The optimal solutions obtained by the Orig. SORA and the ESROA with the constant variances as $\sigma_{x_1,x_2} = 0.01$, $\sigma_{x_4} = 0.02$, $\sigma_{x_5} = 0.03$, $\sigma_{x_6,x_7} = 0.005$ are given in Table 8. The starting points are the same for both methods as $[2.65\ 0.63\ 18\ 6.8\ 6.4\ 3.0\ 5.099]$. The same strategy and condition are adopted for both methods that the convergent criterion is $g_i \le 0,\ i = 1 \sim 11$ and 0.01% for the value of the objective function. From Table 8, the cycles and NFE needed in ESORA are both less than those of the Orig. SORA which indicates that the ESORA is more efficient than the Orig. SORA.

The optimal solutions of Orig. SORA and ESORA with varying variances are provided in Table 9. The starting points are the same for both methods as $[2.65\ 0.63\ 18\ 6.8\ 6.4\ 3.0\ 5.099]$. The convergent criterion is $g_i \le 0,\ i = 1 \sim 11$ and 0.01% for the value of objective function. For both cases of $r_i = 0.008$ and $r_i = 0.01$, the cycles and NFE needed in the ESORA are both much less than those of Orig. SORA especially when the value of constant coefficient of variation increases which indicates that the ESORA is much more efficient than the Orig. SORA.

Table 8. Optimal solutions of speed reducer design with constant variances.

Design variables	Orig. SORA	ESORA
μ_{x_1}	3.6000	3.6000
μ_{x_2}	0.6863	0.6863
μ_{x_3}	18	18
μ_{x_4}	7.3000	7.3000
μ_{x_5}	7.9270	7.9269
μ_{x_6}	3.3669	3.3668
μ_{x_7}	5.3023	5.3022
Objective	3087.6573	3087.6572
Constraints		
g_1	-0.0936	-0.0935
g_2	-0.2403	-0.2403
g_3	-0.5154	-0.5153
g_4	-0.8991	-0.8991
g_5	-16.4314	-16.4314
g_6	-7.4804	-7.4803
g_7	-27.9441	-27.9440
g_8	-6.7546	-6.7545
g_9	-0.2454	-0.2454
g_{10}	-0.0479	-0.0478
g_{11}	-0.0245	-0.0245
Cycles	4	3
NFE	3255	2376

5. Conclusions

SORA is one of the most efficient single loop methods for dealing with RBDO. In this paper, an enhanced SORA is proposed with the aim of further improving the computational efficiency considering both cases of constant and varying variances while keeping the single loop framework. In the ESORA, when the performance functions are linear, the original RBDO problem is completely transformed into a deterministic optimization problem. When the performance functions are not all linear, in the deterministic optimization, the gradient at the actual MPP is approximated using the actual MPP

Table 9. Optimal solution of speed reducer design with varying variances.

Design variables	$r_i = 0.008$		$r_i = 0.01$	
	Orig. SORA	ESORA	Orig. SORA	ESORA
μ_{x_1}	3.6000	3.6000	3.6000	3.6000
μ_{x_2}	0.6825	0.6824	0.6871	0.6871
μ_{x_3}	17	17	17	17
μ_{x_4}	7.3000	7.3000	7.3761	7.3760
μ_{x_5}	8.0870	8.0870	8.2000	8.2000
μ_{x_6}	3.4330	3.4329	3.4543	3.4542
μ_{x_7}	5.4169	5.4168	5.4504	5.4504
Objective	3067.7527	3067.7526	3120.6480	3120.6480
Constraints				
g_1	-0.0527	-0.0528	-0.0656	-0.0656
g_2	-0.1797	-0.1796	-0.1908	-0.1908
g_3	-0.5340	-0.5341	-0.5343	-0.5342
g_4	-0.8978	-0.8978	-0.8968	-0.8967
g_5	-77.3238	-77.3238	-96.0708	-96.0708
g_6	-59.7486	-59.7486	-74.2351	-74.2352
g_7	-28.3983	-28.3982	-28.3186	-28.3186
g_8	-6.7249	-6.7249	-6.76088	-6.76089
g_9	-0.2751	-0.2750	-0.2391	-0.2391
g_{10}	-0.0343	-0.0343	-0.0399	-0.0399
g_{11}	-0.0282	-0.0282	-0.0371	-0.0371
Cycles	8	4	5	4
NFE	8796	4412	5280	4271

and the mean values of random variables of previous cycle, and the mean values of random variables of current cycle while the gradient is approximated at the mean value of the random design variables and parameters at the first cycle.

As demonstrated by the examples, the ESORA performs more efficiently compared with the current approaches when the performance functions are all linear. The cycles needed in ESORA for RBDO with nonlinear constraints are not more than those needed in the compared approaches while the NFE can be reduced using the same starting points and optimization methods in the compared approaches. In the speed reducer deign example, the same starting points and the optimization method are utilized, the cycles and NFE of ESORA are much less than those of the original SORA especially when the value of constant coefficient of variation increases, which indicates that the ESORA is much more efficient than the original SORA.

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References

[1] S. Kim, S. Jun, H. Kang, Y. Park and D. H. Lee, Reliability based optimal design of a helicopter considering annual variation of atmospheric temperature, *Journal of Mechanical Science and Technology*, 25 (5) (2011) 1095-1104.

[2] Y. J. Park, S. Jun, S. Kim and D. H. Lee, Design optimization of a loop heat pipe to cool a lithium ion battery onboard a military aircraft, *Journal of Mechanical Science and Technology*, 24 (2) (2010) 609-618.

[3] Z. Wang, G. Li, H-Z Huang, X. L. Zhang and Y. Li, Reliability-based design optimization for box-booms considering strength degradation and random total working time, *Journal of Mechanical Science and Technology*, 26 (7) (2012) 2045-2049.

[4] H. Z. Huang, X. Zhang, D. B. Meng, Y. Liu and Y. F. Li, Multidisciplinary design optimization with discrete and continuous variables of various uncertainties, *International Journal of Computational Intelligence Systems*, 5 (1) (2012) 93-110.

[5] H. Z. Huang, H. Yu, X. Zhang, S. Zeng and Z. Wang, Collaborative optimization with inverse reliability for multidisciplinary systems uncertainty analysis, *Engineering Optimization*, 42 (8) (2010) 763-773.

[6] H. Z. Huang and X. Zhang, Design optimization with discrete and continuous variables of aleatory and epistemic uncertainties, *Journal of Mechanical Design*, 131 (3) (2009) 031006-1-031006-8.

[7] J. Tu and K. K. Choi, A new study on reliability based design optimization, *Journal of Mechanical Design*, 121 (4) (1999) 557-564.

[8] B. D. Youn and K. K. Choi, Selecting probabilistic approaches for reliability-based design optimization, *AIAA Journal*, 42 (1) (2004) 124-131.

[9] B. D. Youn, K. K. Choi and L. Du, Enriched performance measure approach for reliability based design optimization, *AIAA Journal*, 43 (4) (2005) 874-884.

[10] B. D. Youn, K. K. Choi and Y. H. Park, Hybrid analysis method for reliability-based design optimization, *Journal of Mechanical Design*, 125 (2) (2003) 221-232.

[11] B. D. Youn, K. K. Choi and L. Du, Adaptive probability analysis using an enhanced hybrid mean value method, *Structural and Multidisciplinary Optimization*, 29 (2) (2005) 134-148.

[12] X. Du and W. Chen, A most probable point based method for uncertainty analysis, *Proc. of ASME 2000 DETC/CIE*, Baltimore, Maryland, USA (2000) DETC2000/DAC-14263.

[13] Y. T. Wu, Y. Shin, R. H. Sues and M. A. Cesare, Safety factor based approach for probability based design optimization, *Proc. of 42nd AIAA/ASME/ASC/AHS/ASC SDM Conference and Exhibition Seattle*, Washington, USA (2001) AIAA-2001-1522.

[14] X. Du and W. Chen, Sequential optimization and reliability

assessment method for efficient probabilistic design, *Journal of Mechanical Design*, 126 (2) (2004) 225-233.
[15] T. M. Cho and B. C. Lee, Reliability-based design optimization using convex approximations and sequential optimization and reliability assessment method, *Journal of Mechanical Science and Technology*, 24 (1) (2010) 279-283.
[16] X. L. Yin and W. Chen, Enhanced sequential optimization and reliability assessment method for probabilistic optimization with varying design variance, *Structures and Infrastructure Engineering*, 2 (3-4) (2006) 261-275.
[17] J. Liang, Z. P. Mourelatos and J. Tu, A single-loop method for reliability-based design optimization, *Proc. of ASME 2004 IDETC/CIE*, Salt Lake City, UT, USA (2004) 419-430.
[18] S. Shan and G. G. Wang, Reliability design space and complete single-loop reliability-based design optimization, *Reliability Engineering & System Safety*, 93 (8) (2008) 1218-1230.
[19] R. Rackwitz and B. Flessler, Structural reliability under combined random load sequences, *Computers & Structures*, 9 (5) (1978) 489-494.
[20] L. Du and K. K. Choi, An inverse analysis method for design optimization with both statistical and fuzzy uncertainties, *Structural and Multidisciplinary Optimization*, 37 (2) (2008) 107-119.

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