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Comparisons of Two Non-probabilistic Structural Reliability Analysis Methods for Aero-engine Turbine Disk

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Abstract: Turbine disk is a key component of aero-engine and the failure of turbine disk will lead to disastrous consequences, making the structural reliability analysis for the turbine disk as an urgent issue. Taking the turbine disk as the case study, this paper will compare two non-probabilistic structural reliability analysis methods of imprecise structural reliability analysis and interval structural reliability analysis aiming at providing a more profound understanding about the theoretical system of imprecise probability theory. Moreover, according to the comparisons, this paper will predict the prospects or the works should to be done for the widely application of imprecise probability theory.

Keywords: imprecise probability theory, interval analysis, structural reliability analysis, aero-engine, turbine disk

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1 Introduction

There are more and more stringent requirements on the safety of aero-engines due to the growing technical and environmental complexity. This has stimulated the research and development of reliability analysis methods and assessment procedures for aero-engines [1]. Turbine disk is a key component of aero-engine and the failure of turbine disk may lead to disastrous consequences, thus structural reliability analysis for the turbine disk is an urgent issue. In accordance with the theory of mathematical statistics, structural reliability analysis method can be classified as probabilistic structural reliability analysis method and non-probabilistic structural reliability analysis method [2]. Probabilistic structural reliability analysis method adopts probability theory to quantify information about uncertainty aspects and it is based on deterministic parameters and models. Specifically, strength, stiffness, geometry, loading as well as other characteristic parameters relevant to structural reliability are described by a set of stochastic variables that are grouped into one vector. Then a particular probability distribution is assigned to each of these variables [3]. As we know, probability theory is constructed based on a massive statistical data, hence probabilistic structural reliability analysis methods are very effective when enough statistical data related to structural system are available. However, the problem of lack of data is often encountered in the process of structural reliability analysis of complex systems [4–6]. For example we may not have enough data or information to determine the influence of spatial temperature variation on the reliability of artificial satellite in the early stage of the development. In the case of the turbine disk, its structure, loading conditions as well as working environments are very complicated. Under the cases structural reliability analysis models as well as their parameters are difficult to be uniquely determined. Meanwhile it is unreliable to consider any specific distribution for the variable [7–9]. Probabilistic structural reliability analysis methods are no longer applicable for these systems with small samples or few statistical data. Thus, non-probabilistic structural reliability analysis methods are proposed.

Structural reliability analysis methods based on interval analysis [10], Dempster-Shafer theory [11], fuzzy set theory [12, 13] are collectively known as non-probabilistic structural reliability analysis method. All of these non-probabilistic structural reliability analysis methods assume reliability indexes can be quantified with a bound rather than using a deterministic value. Interval structural reliability analysis assumes unknown parameters of limit state function vary in intervals and adopts

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a non-probabilistic structural reliability index η to characterize structural reliability [10]. Dempster-Shafer theory is a mathematical approach to dealing with evidence coming from different sources and arriving at a degree of belief which is represented as a belief function [11]. Fuzzy reliability theory considers the probabilities of variables can be treated as fuzzy numbers and possibility can be used to quantify the uncertainty instead of probability measures [12, 13]. In addition to non-probabilistic structural reliability analysis method, structural reliability analysis based on imprecise probability theory (so called as imprecise structural reliability analysis) is proposed in recent years and it has received much attentions. The mathematical theory of imprecise probability theory is established on the basis of behavioral interpretation and the imprecise reliability models are constructed by considering the upper and lower previsions of gambles [14]. Regardless whether the method is based on interval analysis, Dempster-Shafer theory, fuzzy set theory or imprecise probability theory, all non-probabilistic structural reliability analysis methods have to balance the quantification of objective uncertainties and the quantification of subjective uncertainties. Study on imprecise structural reliability analysis is still in its early stage, related theories and methods are not mature and engineering applications are rarely reported. Despite all this, the advantages of imprecise probability theory have gained the popularity in the research community. Among these non-probabilistic structural reliability analysis methods, imprecise structural reliability analysis method shares a lot of similarities with interval structural reliability analysis method. In this regard, taking the turbine disk as the study object, this paper will make a comparison of these two non-probabilistic structural reliability analysis methods aiming at providing a much more profound understanding about the theoretical system of imprecise probability theory and predicting the prospects or the works should to be done for the widely application of imprecise probability theory.

The remainder of this article is organized as follows. Some basic knowledge of imprecise structural reliability analysis and interval structural reliability analysis are respectively given in Sections 2 and 3. Immediately following, detailed comparisons of the two non-probabilistic structural reliability analysis methods on aspects such as modeling ideas, model structures, precision, etc. are introduced in Section 4. In Section 5 we consider non-probabilistic structural reliability analysis for the turbine disk as the project example to verify the comparisons. Conclusions are dawn in Section 6.

2 Imprecise structural reliability analysis method

2.1 Basic concepts of imprecise probability theory

The theoretical framework of imprecise probability theory is proposed by Walley [14]. As we know, in probability theory all probability distributions are constructed on the discussion of events. Similarly, imprecise probability models are constructed on the discussion of 'gambles' which are equal to the 'events' in classical probability theory. According to [15], a gamble is a bounded real-valued function defined on domain Ω. The gamble can be interpreted as a reward whose value depends on the uncertain state ω_i, $i=1, \ldots, n$. If you accept the gamble X, then at some later time the true state ω_k will be determined and you will receive the reward $X(\omega_k)$ in units of utility. Three fundamental principles, that are avoiding sure loss, coherence and natural extension, are constructed by considering subjective rationality. Here, we just give some basic definitions which are quoted in the field of reliability engineering and for more detailed information please refer [14].

Definition 2.1.1 $[\underline{P}(X), \overline{P}(X)]$ are lower and upper probabilities of gamble X. A lower probability $\underline{P}(X)$ is the infimum of all values $P(X)$ may take while the upper probability $\overline{P}(X)$ is the supremum of the same gamble. It's worth mentioning that the lower and upper probabilities should be correlative because they are determined under same situations. As we can see, the lower and upper probabilities can just represent the probabilities of one gamble, in order to represent more reliability indexes, lower and upper probabilities are generalized to upper and lower expectations $[\underline{M}(X), \overline{M}(X)]$ which represent an 'average' of some gambles, that is, $M(X)=E(g(X))$. Actually, many reliability indexes can be rewritten as an expectation of one gamble such as

$$R(t) = \Pr\{X - t \geq 0\} = E\left[I_{[t,+\infty)}(X)\right] = \int_\Omega I_{[t,+\infty)}(X)\rho(X)\mathrm{d}X \quad (1)$$

$$F(t) = P(X \leq t) = \int_{R+} I_{[0,t]}(X)\rho(X)dX = E\left(I_{[0,t]}(X)\right) \quad (2)$$

$$MTTF = \int_{R^+} X\rho(X)dX = E(X) \quad (3)$$

and structural reliability

$$R = E\left(I_{[t,+\infty]}(g(r,s))\right) \tag{4}$$

Definition 2.1.2 Natural extension is one of the key concepts of imprecise probability theory for constructing models to draw inferences from the existing data. In current papers, natural extension models are represented by different forms of optimization models [16] and in which reliability indexes of interest are expressed as objective functions and the existing data make up the feasible region. Assume the unknown reliability index M of interest can be expressed as $M = \mathrm{E}(g(X))$ where $X = (X_1, \ldots, X_n)$ and the available reliability data can be expressed in forms of $\mathrm{E}\left(\varphi_{ij}(x_i)\right)$, then natural extension model of its primal form can be written as

$$\underline{M}(\overline{M}) = \min_P\left(\max_P\right)\int_{\mathrm{R}_+^n} g(X)\rho(X)\mathrm{d}X \tag{5}$$

subject to

$$\begin{gathered}\rho(X)\geq 0,\ \int_{\mathrm{R}_+^n}\rho(X)\mathrm{d}X = 1\\ \underline{a}_{ij}\leq\int_{\mathrm{R}_+^n}\varphi_{ij}(x_i)\rho(X)\mathrm{d}X\leq\overline{a}_{ij}, i\leq n, j\leq m_i\end{gathered} \tag{6}$$

where the set P is all possible n-dimensional density functions $\{\rho(X)\}$ satisfying constraints which are made up by the available reliability data. The essence of natural extension model is to find the supremum and infimum of unknown indexes in the feasible region. Obviously, the available reliability data can be thought as the evidence to reduce the range of set P. If the parameters are independent, $\rho(X)$ can be written as

$$\rho(X) = \rho(x_1)\times\rho(x_2)\times\cdots\times\rho(x_n) \tag{7}$$

Natural extension in its primal form is sometimes too complex to solve, thus Kuznetsov [16] applied the duality theorem of linear programming in eqs (5) and (6) to obtain the new form called Kuznetsov's form. According to the duality theorem, eqs (6) and (7) can be written as:

$$\underline{\mathrm{M}}(g) = \sup_{c, c_{ij}, d_{ij}}\left\{c + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(c_{ij}\underline{a}_{ij} - d_{ij}\overline{a}_{ij}\right)\right\} \tag{8}$$

subject to

$$c + \sum_{i=1}^{n}\sum_{j=1}^{m_i}(c_{ij} - d_{ij})\varphi_{ij}(x_i)\leq g(X) \tag{9}$$

and

$$\overline{\mathrm{M}}(g) = \inf_{c, c_{ij}, d_{ij}}\left\{c + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(c_{ij}\overline{a}_{ij} - d_{ij}\underline{a}_{ij}\right)\right\} \tag{10}$$

subject to

$$c + \sum_{i=1}^{n}\sum_{j=1}^{m_i}(c_{ij} - d_{ij})\varphi_{ij}(x_i) >= g(X) \tag{11}$$

Here c, c_{ij}, d_{ij} are optimization variables, c corresponds to the constraint condition $\int_{\mathrm{R}_+^n}\rho(X)\mathrm{d}X = 1$, c_{ij} corresponds to the constraint condition $\int_{\mathrm{R}_+^n}\varphi_{ij}(x_i)\rho(X)\mathrm{d}X\leq\overline{a}_{ij}$, and d_{ij} corresponds to the constraint condition $\int_{\mathrm{R}_+^n}\varphi_{ij}(x_i)\rho(X)\mathrm{d}X\geq\underline{a}_{ij}$.

2.2 Imprecise structural reliability analysis

Imprecise structural reliability analysis is based on the assumption that the available information can be expressed in form of upper and lower expectations [2, 3]. The problem of structural reliability analysis based on imprecise probability theory can be stated as follows. Let limit state function of a structural system be denoted by $g(X)$ where $X = (X_1, X_2, \ldots, X_n)$ and $X = (X_1, X_2, \ldots, X_n)$ is a vector of all physical variables related to structural reliability, such as load, strength, temperature, impact value, material characteristics and others. Failure region of this structural system can be expressed as Φ where $\Phi = \{X : g(X) < 0\}$. So, the reliability of this structure can be computed as $R = \Pr\{g(X)\geq 0\}$ and the failure probability can be computed as $F = \Pr\{g(X) < 0\} = 1 - R$.

Assume that there are m statistical data collected for vector X and they can be expressed as upper and lower expectations of gambles $f_i(X)$, that is $[\overline{a}_i, \underline{a}_i]$, and $f_i(X)$ are known real-valued functions of variable X. Thus, the statistical data can be written as

$$\underline{a}_i\leq\int_\Omega f_i(X)\rho(X)\mathrm{d}X\leq\overline{a}_i \tag{12}$$

where $\rho(X)$ is the joint probability density function.

According to natural extension model, structural reliability can be rewritten in form of $R = \mathrm{E}\left(I_{[0,+\infty)}(g(X))\right)$. Then, the lower and upper bound of structural reliability can be computed by using the following equations.

$$\begin{aligned}\underline{R} &= \inf_P\int_\Omega I_{[0,+\infty)}(g(X))\rho(X)\mathrm{d}X\\ \overline{R} &= \sup_P\int_\Omega I_{[0,+\infty)}(g(X))\rho(X)\mathrm{d}X\end{aligned} \tag{13}$$

subject to

$$\rho(X) \geq 0, \int_{\Omega} \rho(X)\mathrm{d}X = 1,$$
$$\underline{a}_i \leq \int_{\Omega} f_i(X)\rho(X)\mathrm{d}X \leq \overline{a}_i, i = 1, \ldots, m \tag{14}$$

where P is the set of all possible probability density functions $\rho(X)$ satisfying constraints (14).

3 Interval structural reliability analysis

Interval analysis method is also an effective method to deal with the situation in lack of data. Assume that every x_i in vector $X = (x_1, x_2, \ldots, x_n)$ varies in an interval $x_i^I = [\underline{x}_i, \overline{x}_i]$ and $X = (x_1, x_2, \ldots, x_n)$ is the same physical parameter vector in Section 2.2. According to interval analysis, the midpoint and radius of the vector X can be calculated by

$$X^c = \left(x_1^c, x_2^c, \ldots, x_n^c\right), X^r = \left(x_1^r, x_2^r, \ldots, x_n^r\right) \tag{15}$$

Therefore, X^I and x_i can be expressed in the following standard

$$X^I = X^c + X^r\Delta^I, x_i = x_i^c + x_i^r\delta_i \tag{16}$$

where $\Delta^I = [-1, 1]$ is called the standardized unit interval and $\delta_i \in \Delta^I$ is called the standardized unit interval variable.

For interval variables X and Y, the four basic algebraic operations commonly used in literatures includes

$$\begin{aligned}
\left[\underline{X+Y}, \overline{X+Y}\right] &= \left[\underline{x}+\underline{y}, \overline{x}+\overline{y}\right] \\
\left[\underline{X-Y}, \overline{X-Y}\right] &= \left[\underline{x}-\overline{y}, \overline{x}-\underline{y}\right] \\
\left[\underline{X \cdot Y}, \overline{X \cdot Y}\right] &= \left[\min \underline{x}\cdot\underline{y}, \underline{x}\cdot\overline{y}, \overline{x}\cdot\underline{y}, \overline{x}\cdot\overline{y},\right. \\
&\qquad \left.\max \underline{x}\cdot\underline{y}, \underline{x}\cdot\overline{y}, \overline{x}\cdot\underline{y}, \overline{x}\cdot\overline{y}\right] \\
\left[\underline{X/Y}, \overline{X/Y}\right] &= \left[\left(\underline{X}, \overline{X}\right) \cdot \left(1/\overline{Y}, 1/\underline{Y}\right), 0 \notin \left[\underline{Y}, \overline{Y}\right]\right]
\end{aligned} \tag{17}$$

As mentioned above, the limit state function for a structure system can be written as

$$M = g(X), X = (X_1, X_2, \ldots, X_n), X_i \in X_i^I \tag{18}$$

If $g(X) > 0$, the structure is safe, if $g(X) < 0$, the structure fails, and if $g(X) = 0$, the structure reaches its limit state. Because X is an interval variable, then M is an interval variable which has midpoint M^c and radius M^r. When limit state function is a continuous function, the non-probabilistic structural reliability index can be expressed as [17]

$$\eta = \frac{M^c}{M^r} \tag{19}$$

Obviously, if $\eta \geq 1$, this structure is reliable, if $\eta \leq -1$, this structure is unreliable, and if $-1 \leq \eta \leq 1$, both reliable and failed are possible. While the value of η is bigger, the reliability level is higher.

If limit state function $g(X)$ is a linear function, we can easily get the range of M by algebraic operations. However, eq. (18) is usually a nonlinear function with uncertainty parameters which have small deviations, so we have to expand eq. (18) by first order Taylor expansion at X^c. The expansion process can be expressed as [18]

$$M = g(X^c + X^r\Delta) = g(X^c) + \sum_{i=1}^{n} \left.\frac{\partial g(X)}{\partial X_i}\right|_{X^c} X^r\delta_i \tag{20}$$

So, non-probabilistic structural reliability index can be computed by

$$\eta \approx \frac{g(X^c)}{\sum_{i=1}^{n} \left.\frac{\partial g(X)}{\partial X_i}\right|_{X^c} X^r} \tag{21}$$

However, when limit state function is a highly nonlinear function, error owing to approximation in eq. (21) is very large. So optimization method which is more robust and efficient for structural reliability analysis is proposed by Guo [18]. According to Guo, the upper and lower bound of M can be computed by the following optimization model

$$M^U = \max g(X), M^L = \min g(X) \tag{22}$$

subject to

$$X^L \leq X \leq X^U \tag{23}$$

So the non-probabilistic reliability index can be computed by

$$\eta = \frac{M^c}{M^r} = \frac{M^U + M^L}{M^U - M^L} \tag{24}$$

4 Comparisons of imprecise structural reliability analysis and interval structural reliability analysis

In this section, firstly we will introduce how non-probabilistic structural reliability index η can be approximately calculated by imprecise structural reliability analysis method. Then, we will make a comparison of the two non-probabilistic structural reliability analysis methods on aspects such as modeling ideas, model structures, and precision.

4.1 Non-probabilistic structural reliability index' approximately computation

4.1.1 Independent variables

Assume that variables X_i are independent and they are interval variables. Here, we adopt optimization model to compute the non-probabilistic structural reliability index η, so the limit state function as well as its non-probabilistic structural reliability index are

$$\begin{aligned} &M = g(X), X = (X_1, \ldots, X_n) \\ &\eta = \frac{M^c}{M^r} = \frac{M^U + M^L}{M^U - M^L} \end{aligned} \tag{25}$$

where M^U, M^L are computed by eqs (22) and (23).

In order to make a comparison with imprecise structural reliability analysis method, we transfer eq. (16) in the following form

$$X \in X^I = [X^c - X^r, X^c + X^r] \tag{26}$$

Now we have no information about the probabilities or probability distributions of variables X_i and we only know that variables X_i vary in intervals X_i^I, that is to say, variable X_i may take any value of intervals X_i^I. As mentioned above, natural extension models can just deal with information which can be expressed as upper and lower expectations. For a comparison, here, let $\underline{M}^\circ$ and $\overline{M^\circ}$ represent average limit state function's upper and lower bound which can be expressed as upper and lower expectations of $g(X)$, that is $\underline{M}^\circ = \underline{\mathrm{E}}(g(X))$ and $\overline{M^\circ} = \overline{\mathrm{E}}(g(X))$. In the following sections, variables with superscript "$^\circ$" mean these variables are applied or computed in imprecise structural reliability analysis.

Here, variables are considered to be independent. According to natural extension model the upper and lower bound of average structural limit state function can be computed by

$$\begin{aligned} &\underline{M}^\circ = \inf \int_{x_1^c - x_1^r}^{x_1^c + x_1^r} \cdots \int_{x_n^c - x_n^r}^{x_n^c + x_n^r} g(X)\rho(X_1)\cdots\rho(X_n)\mathrm{d}X_1\cdots\mathrm{d}X_n \\ &\overline{M^\circ} = \sup \int_{x_1^c - x_1^r}^{x_1^c + x_1^r} \cdots \int_{x_n^c - x_n^r}^{x_n^c + x_n^r} g(X)\rho(X_1)\cdots\rho(X_n)\mathrm{d}X_1\cdots\mathrm{d}X_n \end{aligned} \tag{27}$$

subject to

$$\rho(X_i) \geq 0, \int_{x_1^c - x_1^r}^{x_1^c + x_1^r} \rho(X_1)\mathrm{d}X_1 = 1, \ldots, \int_{x_n^c - x_n^r}^{x_n^c + x_n^r} \rho(X_n)\mathrm{d}X_n = 1 \tag{28}$$

and the non-probabilistic structural reliability index under the case of independent variables can be calculated by

$$\eta^\circ_{Independent} = \frac{M^{\circ c}}{M^{\circ r}} = \frac{\overline{M}^\circ + \underline{M}^\circ}{\overline{M}^\circ - \underline{M}^\circ} \tag{29}$$

In this way interval structural reliability analysis models with dependent variables are transformed to imprecise structural reliability analysis models.

4.1.2 Dependent variables

For dependent variables, non-probabilistic structural reliability index can also be approximately computed by imprecise structural reliability model. Assume that dependent variables X_i vary in intervals X^I. In this case, the bound of structural performance can be computed by

$$\begin{aligned} &\underline{M}^\circ = \inf \int_{x_1^c - x_1^r}^{x_1^c + x_1^r} \cdots \int_{x_n^c - x_n^r}^{x_n^c + x_n^r} g(X)\rho(X_1, \ldots, X_n)\mathrm{d}X_1\cdots\mathrm{d}X_n \\ &\overline{M^\circ} = \sup \int_{x_1^c - x_1^r}^{x_1^c + x_1^r} \cdots \int_{x_n^c - x_n^r}^{x_n^c + x_n^r} g(X)\rho(X_1, \ldots, X_n)\mathrm{d}X_1\cdots\mathrm{d}X_n \end{aligned} \tag{30}$$

subject to

$$\rho(X_i) \geq 0, \int_{x_1^c - x_1^r}^{x_1^c + x_1^r} \cdots \int_{x_n^c - x_n^r}^{x_n^c + x_n^r} \rho(X_1, \ldots, X_n)\mathrm{d}X_1\cdots\mathrm{d}X_n = 1 \tag{31}$$

and the non-probabilistic structural reliability index can be calculated by

$$\eta^\circ = \frac{M^{\circ c}}{M^{\circ r}} = \frac{\overline{M}^\circ + \underline{M}^\circ}{\overline{M}^\circ - \underline{M}^\circ} \tag{32}$$

Here, the constraints (31) make up a very large feasible region thus the result $\left[\underline{M}^\circ, \overline{M}^\circ\right]$ is also very large and imprecise.

As we know, non-probabilistic structural reliability index is sensitive to its midpoint and radius of interval variables, more or less [19]. In this model, η° is more sensitive to $M^{\circ r}$, $M^{\circ r} = \overline{M^\circ} - \underline{M}^\circ$, because $M^{\circ r}$ under partial information is usually very large especially in the case of dependent variables. In this case, approaches to decrease the imprecision of non-probabilistic structural reliability index should be introduced. Actually, the dependence relationship of dependent variables can be quantified by

Copula function. According to [20], joint probability density function can be transformed into a product of Copula function and its marginal probability density functions. And if marginal probability density functions are continuous, Copula function can be only determined. So the introduction of Copula function can be as an additional constraint to eq. (31) so that the feasible region will be reduced. Here, we introduce n-dimensional Copula function to represent the relationship of variables in eqs (30) and (31), so the average structural performance can be rewritten as

$$M^{\circ} = E(g(X)) = \int_{\Omega} g(X) \frac{\partial^n C_{\theta}(u_1, \cdots, u_n)}{\partial u_1 \cdots \partial u_n}\bigg|_{u_1 = \rho_1(x), \cdots, u_n = \rho_n(x)} \rho(X_1) \cdots \rho(X_n) \mathrm{d}X_1 \cdots \mathrm{d}X_n \tag{33}$$

where $\rho_i(\cdot)$ are marginal probability density functions of variables $X_1, \ldots, X_n$ and $C_{\theta}(u_1, \ldots, u_n)$ characterizes the relationship of $\rho_i(\cdot)$, and θ is relevant parameter.

So the upper and lower bound of average structural performance can be rewritten as

$$\begin{aligned} \underline{M^{\circ}} &= \inf \int_{x_1^c - x_1^r}^{x_1^c + x_1^r} \cdots \int_{x_n^c - x_n^r}^{x_n^c + x_n^r} g(X) \frac{\partial^n C_{\theta}(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n}\bigg|_{u_1 = \rho_1(x), \ldots, u_n = \rho_n(x)} \rho(X_1) \cdots \rho(X_n) \mathrm{d}X_1 \cdots \mathrm{d}X_n \\ \overline{M^{\circ}} &= \sup \int_{x_1^c - x_1^r}^{x_1^c + x_1^r} \cdots \int_{x_n^c - x_n^r}^{x_n^c + x_n^r} g(X) \frac{\partial^n C_{\theta}(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n}\bigg|_{u_1 = \rho_1(x), \ldots, u_n = \rho_n(x)} \rho(X_1) \cdots \rho(X_n) \mathrm{d}X_1 \cdots \mathrm{d}X_n \end{aligned} \tag{34}$$

subject to

$$\rho(X_i) \geq 0, \int_{x_1^c - x_1^r}^{x_1^c + x_1^r} \cdots \int_{x_n^c - x_n^r}^{x_n^c + x_n^r} \frac{\partial^n C_{\theta}(u_1, \ldots, u_n)}{\partial u_1 \ldots \partial u_n}\bigg|_{u_1 = \rho_1(x), \ldots, u_n = \rho_n(x)} \rho(X_1) \cdots \rho(X_n) \mathrm{d}X_1 \cdots \mathrm{d}X_n = 1 \tag{35}$$

4.2 Comparison on the two non-probabilistic structural reliability models

As analyzed above, non-probabilistic reliability index can be approximately calculated by imprecise probabilistic model. So it is necessary to make a comparison of these two models on aspects such as assumptions, modeling ideas, model structures, and applicable condition. So that we can easily determine which method is more suitable in a certain situation and predict the prospects or the works should to be done for the widely application of interval structural reliability analysis.

(1) Not all interval structural reliability analysis models can be transferred to interval structural reliability analysis model.

In interval structural reliability analysis, variables are considered varying in intervals, that is, $Y_i \in Y_i^I$ and it does not consider the variables' probability densities in their subject intervals. While in imprecise structural reliability analysis, unknown probability density distribution is assumed for every variable and the upper and lower bound of structural performance is computed by optimization models. However, when objective function of interest cannot be expressed as the expectation of known functions, non-probabilistic structural reliability indexes can't be calculated by imprecise structural reliability analysis.

(2) Imprecise structural reliability analysis can complement interval structural reliability analysis sometimes.

Interval structural reliability analysis adopts non-probabilistic reliability index η to characterize the degree of a structure's reliability, when $\eta \geq 1$ this structure is reliable, when $\eta \leq -1$ this structure is unreliable, and when $-1 \leq \eta \leq 1$ this structure is in an uncertain domain, that is, when $-1 \leq \eta \leq 1$, we cannot make sure the reliability level of this structure. In this case, the following imprecise structural reliability analysis model can be used to quantify the structural reliability. This model is constructed on the same assumption as in interval structural reliability analysis, so the results are equivalent.

$$\begin{aligned} \underline{R}^{\circ} &= \inf \int_{y_1^c - y_1^r}^{y_1^c + y_1^r} \cdots \int_{y_n^c - y_n^r}^{y_n^c + y_n^r} I_{[0, +\infty)}(g(Y)) \rho(Y_1, \ldots, Y_n) \mathrm{d}Y_1 \cdots \mathrm{d}Y_n \\ \overline{R}^{\circ} &= \sup \int_{y_1^c - y_1^r}^{y_1^c + y_1^r} \cdots \int_{y_n^c - y_n^r}^{y_n^c + y_n^r} I_{[0, +\infty)}(g(Y)) \rho(Y_1, \ldots, Y_n) \mathrm{d}Y_1 \cdots \mathrm{d}Y_n \end{aligned} \tag{36}$$

subject to

$$\rho(Y_i) \geq 0, \int_{y_1^c - y_1^r}^{y_1^c + y_1^r} \cdots \int_{y_n^c - y_n^r}^{y_n^c + y_n^r} \rho(Y_1, \cdots, Y_n) \mathrm{d}Y_1 \cdots \mathrm{d}Y_n = 1 \tag{37}$$

4.3 Comparison on imprecision

In this section, we will make an imprecision comparison of the two non-probabilistic structural reliability analysis methods.

In imprecise structural reliability analysis, non-probabilistic structural reliability index is calculated by

$$\eta^{\circ} = \frac{M^{\circ c}}{M^{\circ r}} = \frac{\overline{M}^{\circ} + \underline{M}^{\circ}}{\overline{M}^{\circ} - \underline{M}^{\circ}}$$

and $\overline{M^{\circ}}, \underline{M}^{\circ}$ are computed by eqs (27) and (28).

Denote $\eta^{\circ}_{Dependent}$ and $\eta^{\circ}_{Independent}$ are the non-probabilistic structural reliability indexes under the cases of dependent variables and independent variables respectively, Obviously, $\eta^{\circ}_{Dependent} < \eta^{\circ}_{Independent}$. When comparing with interval structural reliability analysis, we can get $\eta^{\circ}_{Dependent} < \eta^{\circ}_{Independent} < \eta$. Specific explanation is as follows. In interval structural reliability analysis, unknown parameters are considered as independent and all possible values of X in X^I have been taken participate in the algebraic operations with the same possibilities. While in imprecise structural reliability analysis, its essence is to find the upper and lower bound of unknown variable in a feasible region made up by all possible probability density functions, that is to say, it consider the possibilities of every value may be taken from its obeyed interval, thus we will get a much more imprecise interval and $\overline{M^{\circ}} - \underline{M}^{\circ} \geq \overline{M} - \underline{M}$.

According to probability theory, the expectation of samples is approximately equal to expectation of the whole, so

$$M^{\circ c} \approx M^c \tag{38}$$

Because $[\underline{M}, \overline{M}]$ is belong to $[\underline{M}^{\circ}, \overline{M^{\circ}}]$, so

$$M^{\circ r} \geq M^r \tag{39}$$

Take eqs (38) and (39) to eq. (16), we can get $M^{\circ c} \approx M^c$ that

$$\eta^{\circ}_{Dependent} < \eta^{\circ}_{Independent} < \eta \tag{40}$$

5 Structural reliability analysis for aero-engine turbine disk

Turbine disk is a key component of aero-engine whose working environment is very hash. The turbine disk usually works under high rotation speed, high temperature, high speed gas flow, high pressure and it also bears cyclic loadings of centrifugal force, thermal stress, aerodynamic force, external excitation and impact as well as dynamic forces, thus structural reliability analysis for the turbine disk is very essential. Here we consider a certain type of turbine disk shown as Figure 1.

Figure 1: A certain of type turbine disk.

For this turbine disk its limit state function can be written as [18]

$$M = \sigma_s S - \frac{C\omega^2}{2\pi} - 2\rho\omega^2 J \tag{41}$$

where σ_s is the limit strength, S is cross-sectional area, C is a constant, ω is the rotation rate, ρ is the mass density, and J is cross sectional moment of inertia.

Assume that parameters $\sigma_s, S, C, \omega, \rho, J$ can be expressed as interval form, just as shown in Table 1 [21].

According to interval analysis, the non-probabilistic structural reliability index can be computed by

$$\eta = \frac{M^c}{M^r} = 1.0951 > 1 \tag{42}$$

Assume we don't know the dependence of parameters then the natural extension model in primal form for computing the upper and lower bound of average structural performance can be written as

Table 1: Data for the parameters $\sigma_s, S, C, n, \rho, J$.

Parameters	X^U	X^L	X^c	X^r
σ_s	1, 285.036 MPa	974.764 MPa	1, 130 MPa	155.036 MPa
S	$6.3264 \times 10^{-3}m^2$	$6.0832 \times 10^{-3}m^2$	$6.2048 \times 10^{-3}m^2$	$0.1216 \times 10^{-3}m^2$
C	5.8904	5.446	5.6682	0.2222
n	12, 834.12 rpm	11, 865.88 rpm	12, 350 rpm	484.12 rpm
ρ	8, 724.512 kg/m^3	7, 755.488 kg/m^3	8, 240 kg/m^3	484.512 kg/m^3
J	$1.286079 \times 10^{-4}m^4$	$1.143235 \times 10^{-4}m^4$	$1.214657 \times 10^{-4}m^4$	$0.071422 \times 10^{-4}m^4$

$$\underline{M}^{\circ}=\inf\int_{974.764}^{1285.036}\int_{6.0832\times10^{-3}}^{6..3264\times10^{-3}}\int_{5.466}^{5.8904}\int_{7755.488}^{8724.512}\int_{1242.592}^{1343.9859}\int_{1.143235\times10^{-4}}^{1.286079\times10^{-4}}\left(\sigma_s S-\frac{C\omega^2}{2\pi}-2\rho_0\omega^2 J\right)\rho(\sigma_s,S,C,\omega,\rho,J)\mathrm{d}(\sigma_s,S,C,\omega,\rho,J)$$

$$\overline{M^{\circ}}=\sup\int_{974.764}^{1285.036}\int_{6.0832\times10^{-3}}^{6..3264\times10^{-3}}\int_{5.466}^{5.8904}\int_{7755.488}^{8724.512}\int_{1242.592}^{1343.9859}\int_{1.143235\times10^{-4}}^{1.286079\times10^{-4}}\left(\sigma_s S-\frac{C\omega^2}{2\pi}-2\rho_0\omega^2 J\right)\rho(\sigma_s,S,C,\omega,\rho,J)\mathrm{d}(\sigma_s,S,C,\omega,\rho,J) \tag{43}$$

subject to

$$\int_{974.764}^{1285.036}\int_{6.0832\times10^{-3}}^{6..3264\times10^{-3}}\int_{5.466}^{5.8904}\int_{7755.488}^{8724.512}\int_{1242.592}^{1343.9859}\int_{1.143235\times10^{-4}}^{1.286079\times10^{-4}}\rho(\sigma_s,S,C,\omega,\rho,J)\mathrm{d}(\sigma_s,S,C,\omega,\rho,J)=1 \tag{44}$$

Also, natural extension model for computing the upper and lower bound of average structural performance can be rewritten Kuznetsov's form:

$$\underline{M}^{\circ}=\sup_{c,c_{ij},d_{ij}}\left(c+\sum_{i=1}^{6}(c_{i1}\underline{a}_{i1}-d_{i1}\overline{a}_{i1})\right) \tag{45}$$

subject to

$$c+\sum_{i=1}^{6}(c_{i1}-d_{i1})\le\sigma_s S-\frac{C\omega^2}{2\pi}-2\rho_0\omega^2 J \tag{46}$$

and

$$\overline{M^{\circ}}=\inf_{c,c_{ij},d_{ij}}\left(c+\sum_{i=1}^{6}(c_{i1}\overline{a}_{i1}-d_{i1}\underline{a}_{i1})\right) \tag{47}$$

subject to

$$c+\sum_{i=1}^{6}(c_{i1}-d_{i1})\ge\sigma_s S-\frac{C\omega^2}{2\pi}-2\rho_0\omega^2 J \tag{48}$$

Thus, non-probabilistic reliability index is calculated by

$$\overset{\circ}{\eta}_{Dependent}=\frac{M^{\circ c}}{M^{\circ r}}=\frac{28367.21+1037.27}{28367.21-1037.27}=1.0759 \tag{49}$$

When all parameters are independent then natural extension model for computing the upper and lower bound of average structural performance can be written as:

$$\underline{M}^{\circ}=\inf_{P}\int\left(\sigma_s S-\frac{C\omega^2}{2\pi}-2\rho_0\omega^2 J\right)\rho(\sigma_s)\rho(S)\rho(C)\rho(\omega)\rho(\rho)\rho(J)\mathrm{d}\sigma_s\mathrm{d}S\mathrm{d}C\mathrm{d}\omega\mathrm{d}\rho\mathrm{d}J$$

$$\overline{M^{\circ}}=\sup_{P}\int\left(\sigma_s S-\frac{C\omega^2}{2\pi}-2\rho_0\omega^2 J\right)\rho(\sigma_s)\rho(S)\rho(C)\rho(\omega)\rho(\rho)\rho(J)\mathrm{d}\sigma_s\mathrm{d}S\mathrm{d}C\mathrm{d}\omega\mathrm{d}\rho\mathrm{d}J \tag{50}$$

subject to

$$\int_{974.764}^{1285.036}\int_{6.0832\times10^{-3}}^{6..3264\times10^{-3}}\int_{5.466}^{5.8904}\int_{7755.488}^{8724.512}\int_{1242.592}^{1343.9859}\int_{1.143235\times10^{-4}}^{1.286079\times10^{-4}}\rho(\sigma_s)\rho(S)\rho(C)\rho(\omega)\rho(\rho)\rho(J)\mathrm{d}\sigma_s\mathrm{d}S\mathrm{d}C\mathrm{d}\omega\mathrm{d}\rho\mathrm{d}J=1 \tag{51}$$

For independent parameters, non-probabilistic reliability index is calculated by

$$\overset{\circ}{\eta}_{Independent}=\frac{M^{\circ c}}{M^{\circ r}}=\frac{24761.13+1112.97}{24761.13-1112.97}=1.0941 \tag{52}$$

Thus, we can see that $\overset{\circ}{\eta}_{Dependent}<\overset{\circ}{\eta}_{Independent}<\eta$ and the calculations are in keeping with our comparisons.

6 Conclusion

This paper provides a detailed introduction of imprecise structural reliability analysis method and interval structural reliability analysis method. Comparisons of these two methods in terms of modeling ideas, model structure, imprecision are also made. It is proved that sometimes interval structural reliability analysis model can be transferred to imprecise structural reliability analysis model when the available reliability data can be expressed in the form of upper and lower expectations of known functions. Besides, the results from imprecise structural reliability analysis methods are much more conservative than those from interval structural reliability analysis. The example of turbine disk illustrated the computational process of these two methods and the calculated results are consistent with the comparisons discussed in the paper. Note that this paper just discussed the similarities and differences of the two non-probabilistic structural reliability analysis method and didn't discuss the fusion of the two methods, so future research will focus on how to fuse interval analysis and imprecise probability theory into one model to deal with the problem of hybrid uncertainties.

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References

1. Li YF, Huang HZ, Zhu SP, Liu Y, Xiao NC. An application of fuzzy fault tree analysis to uncontained events of an areo-engine rotor. Int J Turbo Jet Engines 2012;29:309–15.
2. Utkin LV, Kozine IO. Stress-strength reliability models under incomplete information. Int J Gen Syst 2002;31:549–68.

3. Utkin LV, Kozine IO. Structural reliability modelling under partial source information. In: Langseth H, Lindqvist B (eds.): Proc. of the Third International Conference on Mathematical Methods in Reliability (Methodology and Practice). Trondheim, Norway, 2002:647–50.
4. Li YF, Huang HZ, Liu Y, Xiao N, Li H. A new fault tree analysis method: fuzzy dynamic fault tree analysis. Eksploatacja i Niezawodnosc Maintenance Reliab 2012;14:208–14.
5. Li YF, Mi J, Liu Y, Yang YJ, Huang HZ. Dynamic fault tree analysis based on continuous-time Bayesian networks under fuzzy numbers. Proc Inst Mech Eng Part O, J Risk Reliab 2015;229:530–41.
6. Li YF, Huang HZ, Zhang H, Xiao NC, Liu Y. Fuzzy sets method of reliability prediction and its application to a turbocharger of diesel engines. Adv Mech Eng 2013;2013:7, Article ID 216192.
7. Utkin LV, Coolen FP. Imprecise reliability: an introductory overview. Comput Intell Reliab Eng 2007;40:261–306.
8. Coolen FP. On the use of imprecise probabilities in reliability. Qual Reliab Eng Int 2004;20:193–202.
9. Ben-Haim Y. Robust reliability in the mechanical sciences. Berlin: Springer-Verlag, 1996.
10. Elishakoff I. Discussion on a non-probabilistic concept of reliability. Struct Saf 1995;17:195–99.
11. Zadeh LA. A simple view of the dempster-shafer theory of evidence and its implication for the rule of combination. AI Mag 1986;7:85–90.
12. Cai KY. Parameter estimations of normal fuzzy variables. Fuzzy Sets Syst 1993;55:179–85.
13. Huang HZ, Tong X, Zuo MJ. Posbist fault tree analysis of coherent systems. Reliab Eng Syst Saf 2004;84:141–48.
14. Walley P. Statistical reasoning with imprecise probabilities. London: Chapman and Hall, 1991.
15. Kozine IO, Filimonov YV. Imprecise reliabilities: experiences and advances. Reliab Eng Syst Saf 2000;67:75–83.
16. Utkin LV, Kozine IO. Different faces of the natural extension. 2nd International Symposium on Imprecise Probabilities and Their Applications, Ithaca, New York, 2001.
17. Guo SX, Lu ZZ, Feng YS. A non-probabilistic model of structural reliability based on interval analysis. Chinese J Comput Mech 2001;18:56–60.
18. Guo SX, Zhang L, Li Y. Procedures for computing the non-probabilistic reliability index of uncertain structures. Chinese J Comput Mech 2005;22:227–32.
19. Li GJ, Lu ZZ, Wang P. Sensitivity analysis of nonprobabilistic reliability of uncertain structure. Acta Aeronautet Astronautica Sinica 2011;32:1–7.
20. Tang JY, Zhao YX, Song DL. Static and dynamic models for reliability calculation of stress-strength interference. J Southwest Jiaotong Univ 2010;45:384–88.
21. Xiao NC. Structural reliability methods under stochastic and epistemic uncertainties, PhD D. Chengdu: University of Electronic Science and Technology of China, 2012.